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# Chain-complete posets and directed sets with applications

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## 1. Introduction

Let a poset $P$ be called *chain-complete* when every chain, including the empty chain, has a sup in $P$. Many authors have investigated properties of posets satisfying some sort of chain-completeness condition (see [1], [3], [6], [7], [17], [18], [19], [21], [22]), and used them in a variety of applications. In this paper we study the notion of chain-completeness and demonstrate its usefulness for various applications. Chain-complete posets behave in many respects like complete lattices; in fact, a chain-complete lattice is a complete lattice. But in many cases it is the existence of sup's of chains, and not the existence of arbitrary sup's, that is crucial.

More generally, let $P$ be called *chain $\alpha$-complete* when every chain of cardinality not greater than $\alpha$ has a sup. We first show that if a poset $P$ is chain $\alpha$-complete, then every directed subset of $P$ with cardinality not exceeding $\alpha$ has a sup in $P$. This *sharpens* the known result ([8], [18]) that in any chain-complete poset, every directed set has a sup.

Often a property holds for every directed set *if and only if* it holds for every chain. We show that direct (inverse) limits exist in a category if and only if 'chain colimits' ('chain limits') exist. Since every chain has a well-ordered cofinal subset [11, p. 68], one need only work with well-ordered collections of objects in a category to establish or disprove the existence of direct and inverse limits. Similarly, a topological space is compact if and only if every 'chain of points' has a cluster point. A 'chain of points' is a generalization of a sequence.

Chain-complete posets, like complete lattices, arise from closure operators in a fairly direct manner. Using closure operators we show how to form the *chain-completion* $\bar{P}$ of any poset $P$.

The chain-completion $\bar{P}$ of a poset $P$ is a chain-complete poset with the property that any chain-continuous map from a poset $P$ into a chain-complete poset $Q$ extends uniquely to a chain-continuous map from the completion $\bar{P}$ into $Q$, where by a chain-continuous map we mean one that preserves sup's of chains. If $P$ is already chain-complete, then $\bar{P}$ is naturally isomorphic to $P$. This completion is not the MacNeille

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completion, since in general $\bar{P}$ is not a lattice. However, if $P$ is a lattice, or even directed, then so is $\bar{P}$.

Since our emphasis is on chains, the chain-completion of a poset is more natural for us than the MacNeille completion. Moreover, forming the MacNeille completion may require the addition of many points in cases where chains are well-behaved. Finite posets with least elements (which are obviously chain-complete) may be greatly enlarged in the process of constructing the MacNeille completion. However, in some cases the MacNeille completion adds fewer new points then the chain-completion.

Tarski's fixpoint theorem [24] generalizes to chain-complete posets, i.e., if $F: P \to P$ is an isotone map and $P$ is a chain-complete poset, then the set of fixpoints is a chain-complete poset under the induced order. This sharpens the results of Abian and Brown [1] that every isotone self-map of a chain-complete poset has a fixpoint. Conversely, we show that if every isotone map $F: P \to P$ has a least fixpoint, $P$ is chain-complete. We prove several generalizations and extensions of these results. It is of interest to note that the basic fixpoint theorem does not require the axiom of choice for its proof.

Chain $\omega$-complete posets are useful in Dana Scott's theory of computation (see [7] for references), where $\omega$ is the first infinite ordinal. The emphasis there is on how well certain objects approximate other objects, and not in the existence of joins of arbitrary objects, which in general have no 'natural' meaning. Many of the results in this paper are contained in an unpublished manuscript on the theory of computation completed by the author during the summer of 1973 at the IBM Thomas J. Watson Research Center.

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## 2. Decomposition of directed sets

Throughout this paper a *chain* will mean a totally ordered set (it may be empty), and a *directed set* will mean an ordered set having an upper bound for each finite subset. Directed subsets must be nonempty, since they must contain an upper bound for the empty set.

The following is a sharpened version of Iwamura's Lemma [13] (see [16], [23, p. 98]), which will allow us to prove our basic results about the existence of sup's for directed subsets of chain $\alpha$-complete posets. The proof is similar to the proof in [23] and is given here for completeness.

THEOREM 1. *If $D$ is an infinite directed set, then there exists a transfinite sequence $D_\alpha$, $\alpha < |D|$, of directed subsets of $D$ having the following properties:*

(1) *for each $\alpha$, if $\alpha$ if finite, so is $D_\alpha$, while if $\alpha$ is infinite $|D_\alpha| = |\alpha|$ (thus for all $\alpha$, $|D_\alpha| < |D|$);*

(2) *if $\alpha < \beta < |D|$, $D_\alpha \subset D_\beta$;*

(3) $D = \bigcup_\alpha D_\alpha$.

*Proof.* We denote $|D|$ by $\gamma$. Recall that cardinal numbers are the least ordinals of a given cardinality. Well-order $D$ using $\gamma$ as an index set, i.e., $D = \{x_\alpha\}_{\alpha<\gamma}$. For each finite subset $F \subset D$, let $u_F$ denote an upper bound for $F$ in $D$. Let $D_0 = \{x_0\}$, and let $D_{i+1} = D_i \cup \{y_{i+1}, u_{D_i \cup \{y_{i+1}\}}\}$ where $y_{i+1}$ is the least element of $D - D_i$. Thus for each integer $i$, $D_i$ is a finite directed set of cardinality at least $i$ and if $i<j$, $D_i \subset D_j$. Let $D_\omega = \bigcup_{i<\omega} D_i$. Then $D_\omega$ is directed and $|D_\omega| = \omega$. If $|D| = \omega$, we are done. Otherwise, we proceed as follows. Let $\beta < \gamma$ be an infinite ordinal such that for all $\alpha < \beta$, $D_\alpha$ exists and the sequence $\{D_\alpha\}$ $(\alpha<\beta)$ has the required properties. If $\beta$ is a limit ordinal, we simply let $D_\beta = \bigcup_{\alpha<\beta} D_\alpha$. Clearly, for all $\alpha<\beta$ $D_\alpha \subset D_\beta$, and $|D_\beta| = |\beta|$. Finally, suppose $\beta = \delta+1$. Let $D_{\beta,0} = D_\delta \cup \{y_\beta\}$, where $y_\beta$ is the least element of $D - D_\delta$. Inductively we define $D_{\beta,i+1} = D_{\beta,i} \cup \{u_F \mid F \subset D_{\beta,i},\ F \text{ is finite}\}$. Now let $D_\beta = \bigcup_{i<\omega} D_{\beta,i}$. $D_\beta$ is directed, since any finite subset $S \subset D_\beta$ lies in some $D_{\beta,i}$ and hence has an upper bound in $D_{\beta,i+1}$. If $X$ is an infinite set, the cardinality of the set of all finite subsets of $X$ is equal to $|X|$. Thus for all $i$, $|D_\delta| = |D_{\beta,i}|$. And $|D_\beta| \leqslant \omega \times |D_\delta| = |D_\delta| = |\delta| = |\beta|$. Clearly, $D_\delta \subset D_\beta$. It is now clear that $D = \bigcup_{\alpha<\gamma} D_\alpha$. □

*Remark.* If $L$ is an uncountably infinite *lattice*, we can write $L = \bigcup_\alpha L_\alpha$ where: each $L_\alpha$ is a sublattice of $L$; $\alpha<\beta$ implies $L_\alpha \subset L_\beta$; $|L_\alpha| < |L|$ for all $\alpha$. To see this modify the proof of Theorem 1, so that instead of adding upper bounds of finite subsets of $D_{\beta,i}$ to get $D_{\beta,i+1}$, we add sup's and inf's of all finite subsets of $L_{\beta,i}$ to get $L_{\beta,i+1}$. Note that even for finite $\alpha$, $|L_\alpha|$ may be $\omega$.

COROLLARY 1. *Let $P$ be chain $\gamma$-complete. Then for all directed subsets $D$ of $P$ for which $|D| \leqslant \gamma$, $\sup_P D$ exists.*

*Proof.* Suppose the conclusion is false. Let $D \subset P$ be a directed subset such that: 1) $|D| \leqslant \gamma$, 2) $\sup_P D$ does not exist, and 3) for all directed sets $D' \subset P$ with $|D'| < |D|$, $\sup_P D'$ exists. $D$ cannot be finite. Let $D = \bigcup_{\alpha<|D|} D_\alpha$, where the $D_\alpha$ are as in Theorem 1. Let $C = \{\sup_P D_\alpha\}_{\alpha<|D|}$. Clearly, $C$ is a chain and $\sup_P C$ exists and is clearly the sup of $D$ in $P$. This contradiction proves Corollary 1. □

Before proceeding to the remaining corollaries we wish to introduce some further concepts which we will use throughout this paper.

DEFINITION 1. Let $P$ and $Q$ be posets and $f: P \to Q$ a map (posets are nonempty by definition).

(i) $P$ is *strictly inductive (inductive)* if every *non-empty* chain in $P$ has a sup (an upper bound) in $P$.

(ii) $f$ is *chain-continuous* if for all *non-empty* chains $X \subset P$ such that $X$ has a sup in $P$, $f(\sup_P X) = \sup_Q f(X)$.

(iii) $f$ is *chain-*continuous* if for all chains $X \subset P$ such that $X$ has a sup in $P$, $f(\sup_P X) = \sup_Q f(X)$. Thus if $P$ has a least element 0, and $f$ is chain-*continuous, then $Q$ has a least element 0, and $f(0)=0$.

(iv) $f$ is *sup-preserving (inf-preserving)* if for all $X \subset P$ such that $\sup X$ ($\inf X$) exists in $P$:

$$f(\sup_P X) = \sup_Q f(X) \qquad (f(\inf_P X) = \inf_Q f(X)).$$

*Remark*. Observe that chain-continuous, chain-*continuous, sup-preserving, and inf-preserving maps are all isotone (order-preserving). Also note that $\emptyset$ has a sup (inf) in $P$ if and only if $P$ has a least (greatest) element. Finally, much of the material in the following sections extends easily to quasi-ordered sets. We will not discuss the concepts dual to chain-complete, chain-continuous, etc., and leave it to the reader to draw the obvious inferences.

COROLLARY 2. *In a strictly inductive poset (in particular, in a chain-complete poset) every directed subset has a sup.* □

Note that Corollary 2 appears in [8, p. 33]. It can be proved in the same way as Corollary 1.

COROLLARY 3. *Let $P_1$ and $P_2$ be strictly inductive posets and $f: P_1 \to P_2$ be chain-continuous. If $D \subset P_1$ is directed, then $f(\sup_{P_1} D) = \sup_{P_2} f(D)$.*

*Proof.* If $D$ is finite the conclusion is clearly true. For infinite $D$ the corollary follows from Theorem 1 and transfinite induction. □

The following corollaries may be proven more directly by using the fact that sup, of nonempty finite subsets of the poset in question exist (see [9, p. 15, Th. 2.4], [9, p. 9], and [14, p. 163, H]). Note that Corollary 4 is a special case of Corollary 3.

COROLLARY 4. *Let $L$ be a complete lattice such that for every chain $C \subset L$ and $a \in L$, $a \wedge \sup C = \sup_{x \in C}(a \wedge x)$, then for any directed subset $D \subset L$ and $a \in L$, $a \wedge \sup D = \sup_{x \in D}(a \wedge x)$.* □

COROLLARY 5. *Let $P$ be a chain-complete poset such that every finite subset has a sup, then $P$ is a complete lattice. In particular if $P$ is a lattice and a chain-complete poset, then it is a complete lattice.* □

COROLLARY 6. *A topological space $X$ is compact if and only if each nest of closed non-empty sets has a non-empty intersection.* □

*Notation.* Let $P$ and $Q$ be posets, then $P+Q$ ($P \times Q$, $P \oplus Q$), the *cardinal sum (cardinal product, ordinal sum)* of $P$ and $Q$, is the poset consisting of the disjoint union (Cartesian product; disjoint union) of $P$ and $Q$ and ordered as follows. $a \leqslant b$ if and only if $a$ and $b$ both belong to $P$ or $Q$ and in $P$ or $Q$, $a \leqslant b$. ($(a,b) \leqslant (c,d)$ if $a \leqslant c$ and $b \leqslant d$; $a \leqslant b$ if $a \leqslant b$ in $P+Q$ or $a \in P$ and $b \in Q$.) For more details on these operations see [5].



By $\underline{n}$ we shall mean the ordinal $\{0, 1, \ldots, n-1\}$, where $n$ is a non-negative integer. By $\omega$ we mean the first infinite ordinal, i.e., $\omega = \{0, 1, 2, 3, \ldots\}$. We use $\omega_1$ to denote the first uncountable ordinal. Recall that each ordinal is the set of all ordinals which precede it.

The following is an example of a strictly inductive poset $P$ and a directed subset $D$ such that for *no* chain $C \subset D$, $\sup C = \sup D$. Another such example can be found in [4, Theorem 10].

EXAMPLE 1. Let $P = (\omega_1 \oplus 1) \times (\omega \oplus 1)$. $\omega_1 \oplus 1$ and $\omega \oplus 1$ are both complete lattices, and so is $P$. Let $D = \{(a, b) \in P \mid a \neq \omega_1 \text{ and } b \neq \omega\} = \omega_1 \times \omega$. $D$ is directed ($D$ is in fact a lattice). It is easy to see that $\sup D = (\omega_1, \omega)$ but we claim that $(\omega_1, \omega) \neq \sup \{(x_\alpha, y_\alpha)\}_{\alpha \in \Delta}$ for any chain in $D$.

If $C = \{(x_\alpha, y_\alpha)\}_{\alpha \in \Delta}$ is a countable chain in $D$, i.e., $|C|$ is countable, then $\sup \{(x_\alpha, y_\alpha)\}_{\alpha \in \Delta} = (\sup \{x_\alpha\}_{\alpha \in \Delta}, \sup \{y_\alpha\})_{\alpha \in \Delta} < (\omega_1, \omega)$, since $\omega_1$ is not the sup of any countable chain in $\omega_1$. If $C$ is uncountable, then there exists $n \in \omega$ such that $S_n = \{(x_\alpha, n)_{\alpha \in \Delta} \mid (x_\alpha, n) \in C\}$ is uncountable. We now claim that $S_m = \emptyset$ for $m > n$. Suppose that $(x, m) \in S_m$ for some $m > n$. Then $(x, m)$ is an upper bound for $S_n$, hence $(x, m) \geqslant \sup S_n = (\sup_{(x_\alpha, n) \in S_n} \{x_\alpha\}, n) = (\omega_1, n)$, since any uncountable chain in $\omega_1 \oplus 1$ has $\omega_1$ as sup. This implies that $x = \omega_1$, which is impossible since $x \in \omega_1$. Since $S_m = \emptyset$ for all $m > n$ and $S_n$ is uncountable, $\sup C = (\omega_1, n) \neq (\omega_1, \omega)$. Thus no chain in $D$ has $(\omega_1, \omega)$ as a sup. □

The next example shows that a directed subset of an inductive poset need not even have an upper bound.

EXAMPLE 2. Let $Q = (\omega_1 \times \omega) \cup (\{\omega_1\} \times \omega) \cup (\omega_1 \times \{\omega\})$ with the following ordering $(a, b) < (c, d)$ if and only if $a \neq \omega_1$ and $b \neq \omega$ and $(a, b) < (c, d)$ in $(\omega_1 \oplus 1) \times (\omega \oplus 1)$. Thus all elements of the form $(\omega_1, y)$ $(y \neq \omega)$ and $(x, \omega)$ $(x \neq \omega_1)$ are maximal elements in $Q$, and hence at most one such element can be in any chain. The argument used in Example 1 shows that any chain in $\omega_1 \times \omega$ has an upper bound. Thus $Q$ is an inductive poset, but $\omega_1 \times \omega$ has no upper bound in $Q$. □

## 3. Applications to general topology

In this section we briefly indicate the extent to which the preceding results allow one to substitute chains for directed sets when working with topological spaces.

DEFINITION 2. Let $T$ be a topological space and $D$ a directed set. A *net over $D$* is a map $\Pi: D \to T$. If $D$ is a chain (thus $D \neq \phi$), we call nets over $D$ *chains of points over*

$D$. A *cluster point* $y$ of a net $\Pi$ is any point $y \in T$ such that for any neighborhood $\mathscr{U}$ of $y$, and any $i \in D$, there is some $j \in D$, $j \geqslant i$ such that $\Pi(j) \in \mathscr{U}$. A net $\Pi$, *converges to* $y$ if for any neighborhood $\mathscr{U}$ of $y$ there is some $i \in D$ such that for all $j \geqslant i$ $\Pi(j) \in \mathscr{U}$.

*Remark.* Kelley [14, p. 65], defines nets over quasi-ordered directed sets. Our results hold with either definition.

THEOREM 2. *Let $X$ be a topological space. Every net in $X$ has a cluster point if and only if every chain of points in $X$ has a cluster point.*

*Proof.* The necessity is trivial. The sufficiency follows from Theorem 1 by transfinite induction. More precisely, let $D$ be a directed set such that all nets over directed sets of cardinality less than $|D|$ have a cluster point. $D$ is infinite since all finite nets converge. Let $\Pi: D \to X$ be a net. Decompose $D$ as $\bigcup_{\alpha<|D|} D_\alpha$ using Theorem 1. Let $y_\alpha$ be a cluster point of $\Pi \mid D_\alpha$. It is easy to check that the cluster point $y$ of the chain of points $\{y_\alpha\}_{\alpha<|D|}$ is a cluster point for $\Pi$. $\square$

The following corollary follows immediately from Theorem 2 above and Theorem 2 in [14, p. 136].

COROLLARY. *A topological space $X$ is compact if and only if every chain of points in $X$ has a cluster point.* $\square$

*Remark.* The argument above is very similar to the argument in Bruns [6]. In its *set* family form this result can be traced back to Alexandroff-Urysohn [2].

Example $E$ in [14, p. 77] shows that it is possible for a net to converge to a point, without there being a sequence (an $\omega$-chain of points) converging to that point. This result is not surprising in view of the fact that nets can be of arbitrary cardinality, while sequences are countable. In fact, Example B of [14, p. 76] shows that a chain of points can converge to a given point without there existing a sequence which converges to the point in question. The example used in Theorem 10 of [4] can easily be adapted to give a net in $X - \{p\}$ converging to $p$ such that no chain of points in $X - \{p\}$ converges to $p$.

## 4. Chain-complete categories

We now turn our attention to the question of the existence of inverse and direct limits in a category. In particular, we will show that questions of existence of inverse and direct limits can be settled by examing only those cases in which the underlying directed set is a chain. Since every chain has a well-ordered cofinal subset, we need consider only well-ordered chains. Our terminology is that of [20, Chapter 2]. Since our diagrams will have at most one arrow between any two vertices, we can think of a diagram scheme $\Sigma$ as an ordered pair $(I, M)$ with $M \subset I \times I$ and $d: M \to I \times I$ being the inclusion, so we won't bother with $d$. Thus a diagram in a category $C$ over a diagram scheme $\Sigma$ is an ordered pair of maps $(\phi, \theta)$, such that $\phi: I \to \mathrm{Ob}\, C$, $\theta: M \to \mathrm{Mor}\, C$, and $\theta((a, b)) \in \mathrm{Mor}(\phi(a), \phi(b))$.



If $I' \subset I$, we call the diagram scheme $\Sigma' = (I', M' = M \cap (I' \times I'))$ the *restriction of $\Sigma$ induced by $I'$*. Given a diagram $(\phi, \theta)$ in $\mathscr{C}$ over $\Sigma$ we define its *restriction to $\Sigma'$* to be $(\phi \mid I', \theta \mid M')$.

For the remainder of this section we will only refer to *commutative* diagrams, and hence we use the word diagram to mean commutative diagram.

DEFINITION 3. $P$ be a poset. By the *diagram scheme $P\downarrow$ ($P\uparrow$)* we mean $(P, P_\geqslant)$ $((P, P_\leqslant))$ where $P_\geqslant = \{(x, y) \in P \times P \mid x \geqslant y\}$ $(P_\leqslant = \{(x, y) \in P \times P x \leqslant y\})$.

We use the terms *chain family*, *chain cofamily*, *chain limit*, and *chain colimit*, for inverse family, directed family, inverse limit, and direct limit (respectively) when the underlying directed set is a chain.

We say that a category is *chain complete (chain cocomplete)* if chain limits (colimits) exist for every chain family (cofamily) over an arbitrary *nonempty* chain.

*Remark.* For the remainder of this section we will only discuss inverse families, inverse limits, etc., since it is clear that every result has a dual which can be proved dually. If $(\phi, \theta)$ is a diagram over the diagram scheme $P\downarrow$ and $Q \subset P$, we shall denote the *restriction* of $(\phi, \theta)$ to $Q\downarrow$ also by $(\phi, \theta)$.

THEOREM 3. *A category $\mathscr{C}$ is chain complete if and only if an inverse limit exists for every inverse family in $\mathscr{C}$.*

*Proof.* Since every *nonempty* chain is a directed set, the sufficiency is obvious. Necessity follows from Theorem 1 by transfinite induction, as follows.

An inverse limit exists for any inverse family over a finite directed set. Assume that an inverse limit exists for every inverse family over a directed set with cardinality less than the infinite cardinal $\gamma$. Let $D$ be a directed set of cardinality $\gamma$ and $(\phi, \theta)$ and inverse family over $D\downarrow$.

Using Theorem 1, we write $D$ as $\bigcup_{\alpha<\gamma} D_\alpha$. Let $(X_\alpha, \{f_{\alpha,\lambda}\}_{\lambda \in D_\alpha})$ be an inverse limit for $D_\alpha$, where $f_{\alpha,\lambda} \in \mathrm{Mor}(X_\alpha, \phi(\lambda))$. If $\alpha \leqslant \beta < \gamma$, let $g_{\beta,\alpha} \in \mathrm{Mor}(X_\beta, X_\alpha)$ be the unique morphism such that for all $\lambda \in D_\alpha$, $f_{\alpha,\lambda} = f_{\alpha,\lambda} \circ g_{\beta,\alpha}$. It is easy to see that $F = (\{X_\alpha\}_{\alpha<\gamma}, \{g_{\beta,\alpha}\}_{\alpha \leqslant \beta < \gamma})$ is a chain family over $\gamma\downarrow$. Let $(X, \{h_\alpha\}_{\alpha<\gamma})$ be an inverse limit of $F$, with $h_\alpha \in \mathrm{Mor}(X, X_\alpha)$ for all $\alpha$.

For each $\lambda \in D$, we define $f_\lambda \in \mathrm{Mor}(X, \phi(\lambda))$ as follows. There exists $\alpha_0$ such that for all $\alpha \geqslant \alpha_0$, $\lambda \in D_\alpha$. For any $\alpha \geqslant \alpha_0$, let $f_\lambda = f_{\alpha,\lambda} \circ h_\alpha$. It is easy to see that $f_\alpha$ is well-defined, and that $(X, \{f_\lambda\}_{\lambda \in D})$ is an inverse limit for $(\phi, \theta)$ over $D\downarrow$. $\square$

## 5. Closure operators and chain-complete posets

Chain-complete posets correspond to closure operators in a way that generalizes the correspondence between complete lattices and closure operators (see [5, Ch. 5]). The construction and results in this section appear in more general form in Banaschewski [3].

DEFINITION 4. Let $\gamma$ be a closure operator on $X$ and $T\subset 2^X$. We define *the chain-complete poset generated by* $T$ *and* $\gamma$ (denoted by $\gamma_*(T)$) as follows. Let $\Omega = \{Q\subset\gamma(2^X) \mid \gamma(T)\subset Q$ and for all chains $C\subset Q$, $\gamma(\bigcup_{E\in C} E)\in Q\}$. We define $\gamma^*(T)$ to be $\bigcap_{Q\in\Omega} Q$. It is easy to see that $\gamma^*(T)$ is the least element in $\Omega$.

THEOREM 4. *Let $\gamma$ be a closure operator on $X$ and $T\subset 2^X$. Then:*

(a) *$\gamma^*(T)$ is a chain-complete poset and for any chain $C\subset\gamma^*(T)$, $\sup C=\gamma(\bigcup_{E\in C} E)$;*

(b) *If $D\subset\gamma^*(T)$ is a directed set, $\sup D=\gamma(\bigcup_{E\in D} E)$.*

*Proof.* (a) From the definition of $\gamma^*(T)$, it is clear that for any chain $C$ in $\gamma^*(T)$, $\gamma(\bigcup_{E\in C} E)\in\gamma^*(T)$. If $S\in\gamma^*(T)$ is an upper bound for $C$, $\bigcup_{E\in C}\subset S$, i.e., $\gamma(\bigcup E)\subset\gamma(S)=S$ since $S$ is closed. Thus $\gamma(\bigcup E)=\sup C$.

(b) If $D$ is finite, it has a greatest element $S_0$. Then $\sup D=S_0=\gamma(S_0)=\gamma(\bigcup_{E\in D} E)$. Let $D$ be a directed set such that (b) holds for all directed sets of cardinality less than $|D|$. Using Theorem 1 we decompose $D$ into $\bigcup D_\alpha$. Clearly, $\sup D=\sup(\sup D_\alpha)=\gamma(\bigcup_\alpha \gamma(\bigcup_{E\in D_\alpha} E))$ by (a). But $\gamma(\bigcup_\alpha \gamma(\bigcup_{E\in D} E))_\alpha\subset\gamma(\gamma(\bigcup_{E\in D} E))=\gamma(\bigcup_{E\in D} E)$. The reverse inclusion is obvious. The result now follows by transfinite induction. □

*Remark.* $\gamma^*(2^X)=\gamma(2^X)$ is just the usual lattice of closed sets. Arguing as above it is easy to see that $\gamma(\bigcup_{E\in\Gamma} E)$ is the sup of $\Gamma\subset\gamma(2^X)$. Note that $\inf\Gamma=\bigcap_{W\in\Gamma} W$.

The results above extend with trivial modification to strictly inductive posets.

Theorem 5 ($E$) shows that every chain-complete poset is $\gamma^*(T)$ for appropriate $\gamma$, $X$, and $T$.

## 6. The chain-completion of a poset

The chain-completion of a poset, which we describe below, has some nice extension properties with respect to chain-continuous and chain-*continuous maps. Completions of directed sets and lattices are themselves directed sets and complete lattices (respectively).

*Notation.* Given a poset $P$, we use $Ch(P)$ to denote the set of all chains in $P$. We will use $D(P)$ to denote the set of all *order ideals* of $P$, i.e., subsets of $P$ such that whenever they contain an element they contain all elements less than that element as well.

DEFINITION 5. Let $P$ be a poset and $W\subset P$. Let $H_W=\{S\subset P \mid W\subset S$ and for any chain $C\subset S$ and $x\in P$, if $\sup_P C$ exists and $x\leqslant\sup C$, then $x\in S\}$. We define the *chain-closure* of $W$ to be $\bigcap_{S\in H_W} S$ and denote it by $W^\dagger$.



LEMMA 1. *$\dagger$ is a closure operator.*

*Proof.* $W\subset W^\dagger$, $W^\dagger=W^{\dagger\dagger}$ and if $V\subset W$, it is easy to see that $V^\dagger\subset W^\dagger$. □

DEFINITION 6. Let $P$ be a poset. The *chain-completion*, $\bar{P}$, *of* $P$, is simply $\dagger^*(\mathrm{Ch}(P))$.

Theorem 5 gives some basic properties of the chain-completion of a poset. Theorem 6 gives a universal mapping theorem characterization of the chain-completion of a poset.

THEOREM 5. *Let $P$ be a poset and $f:P\to\bar{P}$ be given by $f(x)=[-,x]\equiv\{y\in P \mid y\leqslant x\}$. The following are true.*

(A) *$f$ is chain-*continuous and for $a$, $b\in P$, $a\leqslant b$ if and only if $f(a)\leqslant f(b)$. In particular, $f$ is injective. Furthermore, $f$ is inf-preserving.*

(B) *If $D\subset P$ is directed, then $D^\dagger\in\bar{P}$.*

(C) *If $P$ is directed, then $\bar{P}$ is a directed set with greatest element $P$ and least element $\emptyset^\dagger$.*

(D) *For all $S\in\bar{P}$, $S=\sup_{\bar{P}}\{T\in f(P) \mid T\subset S\}$, i.e., $f(P)$ is join-dense in $\bar{P}$.*

(E) *If $P$ is chain-complete, $f:P\to\bar{P}$ is an isomorphism.*

*Proof.* (A) Clearly $f$ is well-defined and $a\leqslant b$ if and only if $[-,a]\subset[-,b]$. If $y=\sup_P W$ for some chain $W\subset P$, it is easy to see that $f(y)$ is an upper bound for $f(W)$. Let $A\in\bar{P}$ be any upper bound for $f(W)$. Then $W\subset A$, but since $A^\dagger=A$, $y\in A$, i.e., $f(y)\subset A$.

Suppose that $y=\inf_P X$ for some $X\subset P$. Then $f(y)=\bigcap_{x\in X} f(x)$, since if $z\in\bigcap\bigcap_{x\in X} f(x)$, $z$ is a lower bound for $X$, and $z\leqslant y$. Thus clearly $f(y)=\inf_{\bar{P}} f(X)$, since any $S\subset f(x)$ for all $x\in X$, must lie in $\bigcap_{x\in X} f(x)$.

(B) $f(D)\subset\bar{P}$ is a directed set. By Theorem 4($b$), $\sup_{\bar{P}} f(D)=(\bigcup_{x\in D} f(x))^\dagger=D^\dagger$.

(C) Obvious, since $P^\dagger=P$.

(D) Obvious, since $S=\bigcup_{x\in S} f(x)$.

(E) In view of (A) we need only show that $f$ is surjective. Observe that for any chain $C\subset P$, $f(\sup_P C)=C^\dagger$. Thus $\dagger(\mathrm{Ch}(P))\subset f(P)$.

If $C\subset f(P)$ is a chain, so is $f^{-1}(C)\subset P$, since $f$ is isotone in both directions. Let $y=\sup_P f^{-1}(C)$ ($P$ is chain-complete), then since $f$ is chain-continuous, $f(y)=\sup_{\bar{P}} C\in f(P)$, i.e., by Theorem 4, $(\bigcup_{E\in C} E)^\dagger\in f(P)$. By definition of $\bar{P}$, $\bar{P}\subset f(P)$. □

THEOREM 6. *Let $T$ be a chain-complete poset and $h:P\to T$ be chain-*continuous (chain-continuous but not chain-*continuous). Then there exists a* **unique** *map $\bar{h}:\bar{P}\to T$ such that:*

(1) *$\bar{h}$ is chain-*continuous (chain-continuous);*

(2) *$\bar{h}\circ f=h$. Thus any chain-*continuous (chain-continuous) map factors through $\bar{P}$ and $f:P\to\bar{P}$. As usual, it follows that $\bar{P}$ is unique up to isomorphism.*

*Proof.* Let $\bar{h}:\bar{P}\to T$ be given by $\bar{h}(S)=\sup_T h(S)$. We must first show that $\bar{h}$ is well-defined, i.e., that for $S\in\bar{P}$, $\sup_T h(S)$ exists. Assume first that $h$ is chain-*continuous.

We first note that for all $X\subset P$ and $a\in T$, $a$ is an upper bound for $h(X)$ if and only if $a$ is an upper bound for $h(X^\dagger)$. Since $h(X)\subset h(X^\dagger)$, sufficiency is trivial. Now suppose that $a$ is an upper bound for $h(X)$. It follows that $X\subset h^{-1}([0, a])$. If $C$ is any chain in $h^{-1}([0, a])$ such that $\sup_P C$ exists, then since $h$ is chain-*continuous (thus also isotone), $[-, \sup_P C]\subset h^{-1}([0, a])$. Thus $X^\dagger\subset h^{-1}([0, a])$, whence $h(X^\dagger)\subset[0, a]$ and $a$ is an upper bound for $h(X^\dagger)$.

It now follows that $\sup h(X)$ exists if and only if $\sup h(X^\dagger)$ exists and that if they exist, they are equal.

Let $\mathscr{U}=\{X^\dagger \mid X\subset P$ and $\sup_T h(X^\dagger)$ exists$\}$. If $C\subset\mathscr{U}$ is any chain, $\sup_T\{\sup_T h(E) \mid E\in C\}$ exists, since $T$ is chain-complete, and is equal to $h(\bigcup_{E\in C} E)=h((\bigcup E)^\dagger)$. Thus $(\bigcup E)^\dagger\in\mathscr{U}$. By definition, $\bar{P}\subset\mathscr{U}$ and $\bar{h}$ is well-defined on $\bar{P}$.

Observe that for any chain $C$ in $\bar{P}$, $\bar{h}(\sup_{\bar{P}} C)=\sup_T h((\bigcup_{E\in C} E)^\dagger)=\sup h(\bigcup E)=\sup\{\sup h(E) \mid E\in C\}=\sup\bar{h}(C)$. Thus $\bar{h}$ is chain-*continuous.

Observe that $\bar{h}(f(x))=\bar{h}([-, x])=\sup_T h([-, x])=h(x)$ for all $x\in P$, since $h$ is isotone. Thus $\bar{h}\circ f=h$.

We now only need to show that if $h_1:\bar{P}\to T$ is any chain-*continuous map such that $h_1\circ f=h$, then $h_1=\bar{h}$. Let $\mathscr{U}=\{X\in\bar{P} \mid \bar{h}(X)=h_1(X)\}$. Let $C$ be a chain in $P$. By Theorem 4, $C^\dagger=\sup_{\bar{P}}\{f(x) \mid x\in C\}$. Since $h_1$ is chain-*continuous and $h_1\circ f=h$, we have $h_1(C^\dagger)=\sup_T\{h_1\circ f(x) \mid x\in C\}=\sup_T h(C)=\bar{h}(C^\dagger)$. Thus $(\mathrm{Ch}(P))^\dagger\subset\mathscr{U}$. Let $C$ be a chain in $\mathscr{U}$. By Theorem 4, $S=(\bigcup_{E\in C} E)^\dagger=\sup_{\bar{P}} C$. Since $h_1$ is chain-*continuous, $h_1(S)=\sup_{\bar{P}} h_1(C)=\sup_{\bar{P}}\bar{h}(C)=\bar{h}(S)$. Thus $\bar{P}\subset\mathscr{U}$ and $h_1=\bar{h}$.

We briefly discuss the case where $h$ is chain-continuous but not chain-*continuous. This can only occur when $P$ has a least element, $0_P$, and $h(0_P)\neq 0_T$. All of the above goes through except that one must systematically disallow the empty chain, working instead with $\emptyset^\dagger=\{0_P\}$. Thus one would have the $\sup_T h(X)=\sup_T h(X^\dagger)$, except when $X=\emptyset$. We leave it to the reader to make the necessary modifications. □

*Remarks.* In [17; Theorem 4], the following is established and used to prove that the category of chain-complete posets with chain-*continuous maps is cocomplete (in the sense of Mitchell [20]). Let $A$ and $B$ be posets and $f:A\to B$ isotone, then there exists a chain-complete poset $B_{f^+}$ and isotone $g:B\to B_{f^+}$ such that:

(1) $g\circ f$ is chain-continuous;

(2) for all chain-complete posets $H$ and isotone maps $\alpha_1:A\to H$; $\alpha_2:B\to H$ such that $\alpha_1$ is also chain-*continuous and $\alpha_1=\alpha_2\circ f$, there exists a *unique* chain-*continuous map $h:B_{f^+}\to H$ such that $\alpha_2=h\circ g$. The proof of this fact is similar to the proof of Theorem 6. Note that if we let $A=B$ and $f$ be the identity, $B_{f^+}=\bar{B}$.

Examples 3 and 4 (below) show that $f$ in Theorem 5 need not be sup-preserving. Finally, the following corollary shows how to construct a completion of $P$ which is strictly inductive, but not complete, when $P$ lacks a least element.



COROLLARY. *Let $P$ be a poset without a least element and $h$ any chain-continuous (it is also chain-*continuous) map of $P$ into a strictly inductive poset $T$. Then $P'=\bar{P}-\{0_{\bar{P}}\}$ is a strictly inductive but not complete poset and there exists a unique chain-continuous (actually chain-*continuous) map $h':P'\to T$ such that $h'\circ f=h$.*

*Proof.* If $P'$ were complete, it would have a least element $B$. Since $P$ lacks a least element $0_{\bar{P}}=\emptyset=\emptyset^\dagger$, and $f^{-1}(0_{\bar{P}})=\emptyset$ since $f$ is isotone both ways by Theorem 5. Also by Theorem 5(D), $B=\sup\{T\in f(P) \mid T\subset B\}$. Since $B$ is the least element of $P'$, $B\in f(P)$ which implies that $P$ has a least element since $f$ is isotone both ways and $f^{-1}(0_P)=\emptyset$. Thus $P'$ is strictly inductive, but not chain-complete.

Adjoin a new element $\theta$ to $T$ in such a way that $\theta$ is the least element of the resulting poset $T^*$. Consider the unique chain-*continuous map $\bar{h}:\bar{P}\to T^*$ of Theorem 6, such that $\bar{h}\circ f=h$. Let $h'=\bar{h} \mid P'$. It is easy to see that $h'$ is unique. □

LEMMA 2. *Let $P$ be a lattice and $L\subset P$ a sublattice. Then $L^\dagger$ is an element of $\bar{P}$ and a sublattice of $P$ such that:*

(a) *if $W\subset L^\dagger$, $W$ is countable, and $\sup_P W$ exists, then $\sup_P W\in L^\dagger$;*

(b) *if $W\subset L^\dagger$, $W\neq\emptyset$, and $\inf_P W$ exists, then $\inf_P W\in L^\dagger$.*

*Proof.* $L^\dagger\in D(P)$, hence (b) is true. Furthermore, recall that by definition if $C\subset L^\dagger$ is a chain such that $\sup_P C$ exists, and $y\leqslant\sup_P C$, then $y\in L^\dagger$. Since $L$ is directed, by Theorem 5 (B), $L^\dagger\in\bar{P}$. We now show that $L^\dagger$ is a sublattice of $P$.

Let $H=\{E\subset L^\dagger \mid L\subset E$ and $E$ is a sublattice of $P\}$. $H\neq\emptyset$, since $L\in H$. If $F=\{E_\beta\}_{\beta\in\Delta}$ is a non-empty chain in $H$, $E=\bigcup_{\beta\in\Delta} E_\beta\in H$. By Zorn's lemma there exists a maximal element $G\in H$.

Let $C$ be a chain in $G$ and $x\in P$ be such that $\sup_P C=b$ exists and $x\leqslant\sup C$. We wish to show that $x\in G$. Let $Q=\{y\in P \mid y\leqslant b\vee w$ for some $w\in G\}$. We claim that $Q\in H$. Clearly, $L\subset G\subset Q$. If $a$, $a'\in Q$, trivially $a\wedge a'\in Q$. If $a\leqslant b\vee w_1$ and $a'\leqslant b\vee w_2$, then $a\vee a'\leqslant b\vee(w_1\vee w_2)$. Thus $Q$ is a sublattice. It remains only to show that $Q\subset L^\dagger$. Clearly, $b\vee w=\sup\{\theta\vee w \mid \theta\in C\}\in L^\dagger$ for all $C\subset G$ and $w\in G$. Thus $Q\in H$. By maximality, $G=Q$. But it now follows that $G=L^\dagger$.

Finally, let $W=\{w_1, w_2,\ldots\}\subset L^\dagger$ be a countable subset such that $\sup_P W$ exists. Consider the chain $\{w_1, w_1\vee w_2, w_1\vee w_2\vee w_3,\ldots\}$ in $L^\dagger$. $\sup_P W$ is the sup of this chain, hence $\sup_P W\in L^\dagger$. □

THEOREM 7. *Let $P$ be a lattice. Then $\bar{P}$ is a complete lattice, and the map $f:P\to\bar{P}$ of Theorem 5 is inf-preserving, chain-*continuous and preserves countable joins whenever they exist in $P$. If $W\subset\bar{P}$, then $\inf_{\bar{P}} W=\bigcap_{E\in W} E$ and $\sup_{\bar{P}} W=\{\sup_P F \mid F$ finite and $F\subset\bigcup_{E\in W} E\}^\dagger$.*

*Proof.* We first show if $S\subset P$, $S\neq\emptyset$, then $S\in\bar{P}$ if and only if $S$ is a sublattice of $P$ such that $S^\dagger=S$ (i.e., $S$ is closed). In particular $S$ must satisfy (a) and (b) of Lemma 2. If $S$ is a sublattice of $P$ such that $S^\dagger=S$, then by Lemma 2, $S\in\bar{P}$.

Now let $\mathscr{U}=\{S\in\bar{P} \mid \text{such that } S \text{ is a sublattice of } P\}$. Clearly, $C^\dagger\in\mathscr{U}$ for all chains $C\subset P$. The union of a chain of lattices is a lattice and the †-closure of a lattice is a lattice by Lemma 2. Hence $\mathscr{U}=\bar{P}$.

From Theorem 5 we know that $f$ is inf-preserving and chain-*continuous. Let $W=\{w_1, w_2,\ldots\}\subset P$ be countable such that $\sup_P W$ exists (denote it by $w$). Clearly, $f(w)$ is an upper bound of $f(W)$ in $\bar{P}$. Let $T\in\bar{P}$ be any upper bound of $f(W)$. Then $W\subset T$, and since $T$ is a sublattice of $P$ satisfying (a) of Lemma 2, $w\in T$. Since $T\in D(P)$, $f(w)\subset T$. Thus $f(w)=\sup_P f(W)$.

Let $W=\{W_\gamma\}_{\gamma\in\Delta}\subset\bar{P}$. We claim that $\inf_P W=\bigcap_{\gamma\in\Delta} W_\gamma$. Since † is a closure operator and each $W_\gamma$ us closed, it is easy to see that $(\bigcap_{\gamma\in\Delta} W_\gamma)^\dagger=\bigcap_{\gamma\in\Delta} W_\gamma$. If $\bigcap W_\gamma\neq\emptyset$, then $\bigcap W_\gamma\in\bar{P}$, since it is a sublattice which is closed. If $\bigcap W_\gamma=\emptyset$, again $\bigcap W_\gamma\in\bar{P}$ since $\emptyset=\emptyset^\dagger\in\bar{P}$, since $\emptyset$ is a chain. In any event $\bigcap_{\gamma\in\Delta} W_\gamma\in\bar{P}$, and thus clearly it is equal to $\inf_P W$. Thus $\bar{P}$ is a complete lattice with inf corresponding to set-intersection and the greatest element being $P$.

Let $W$ be as above. We claim that $\sup_P W=\{\sup_P F \mid F \text{ finite and } F\subset W_\lambda\}^\dagger$. If $W_\lambda=\emptyset$, this is trivial. Let $D=\{\sup F \mid F \text{ finite and } F\subset\bigcup W_\lambda\}$. Clearly $D$ is directed by Theorem 5 (B), $D^\dagger\in\bar{P}$. It is the sup of $W$ because any other upper bound, $\mathscr{U}$, of $W$ must be a †-closed sublattice of $P$ containing $W_\lambda$ and hence $D$. Thus $D^\dagger\subset\mathscr{U}$. □

We now show that the map $f$ of Theorem 5 and 6 is not sup-preserving in general.

EXAMPLE 3. Let $P=[(\omega_1\oplus N^-)\times(\omega\oplus N^-)-(N^-\times N^-)]\oplus\mathbf{1}$ where $N^-=\{0,-1,-2,-3,\ldots\}$ and where $P$ has the ordering induced by that of $[(\omega_1\oplus N^-)\times(\omega\oplus N^-)]\oplus\mathbf{1}$. It is straightforward to verify that $P$ is a lattice. Let $A_1=\omega_1\times\omega$ and observe that $I=\sup_P A_1$. From the argument in Example 1, it follows that $A_1^\dagger=A_1$. Consider $\bar{P}$. $A_1\in\bar{P}$, since $A_1$ is directed. Observe that $A_1=\sup_P f(A_1)\subset P=f(I)$. Thus $f$ is not sup-preserving. □

The following example shows that even if $P$ is directed, $f$ need not preserve *finite* sups.

EXAMPLE 4. Let $P=(\omega_1\times\omega)\cup\{(\omega_1, 1), (1,\omega), c\}$ be ordered as follows: $P-\{c\}$ is ordered componentwise; $c$ is a maximal element of $P$; $c\geqslant(x,y)$ if and only if $x=1$ or $y=1$. Let $P'=[P\oplus(\mathbf{1}+\mathbf{1})]\oplus\mathbf{1}$. Observe that $c=\sup_P\{(\omega_1,1),(1,\omega)\}$. However, $f(c)$ and $P-\{c\}$ are both upper bounds for $\{f((\omega_1,1)), f((1,\omega))\}$ in $\bar{P}$. Since $f(c)$ and $P-\{c\}$ are non-comparable, $f(c)\neq\sup_P(f((\omega_1,1)), f((1,\omega))\}$. □

We know that if $P$ is a lattice, then $\bar{P}$ is a lattice. However, if $P$ is a distributive lattice (Boolean algebra), $\bar{P}$ need not be a distributive lattice (Boolean algebra). Example 5 also shows that if $P$ is a modular lattice, $\bar{P}$ need not be modular.

EXAMPLE 5. Let $L$ be the distributive lattice described in [9, pp. 71–72]. It is shown in [9] that $L$ cannot even be strongly embedded in a complete modular lattice. (A strong embedding is an injective sup-preserving and inf-preserving map.)



When we construct $L$ we choose $a$ and $b$ to be countable sets, and consequently so are $A$ and $B$. Now $\bar{L}$ is a complete lattice and the map $f:L\to\bar{L}$ of Theorem 5 is injective, chain-*continuous, inf-preserving, and preserves countable joins. The argument on p. 72 of [9] now shows that $\bar{L}$ contains the five-element nonmodular lattice as a sublattice, hence $\bar{L}$ is nonmodular. □

EXAMPLE 6. Let $X$ be an uncountably infinite set and let $B$ be the Boolean algebra of all finite and cofinite subsets of $X$. Then $\bar{B}$ is not a Boolean algebra. Let $D$ be the directed subset of $B$ consisting of all finite sets. By Theorem 5 (B), $D^\dagger\in\bar{B}$. It is not hard to see that $D^\dagger=D$. Any complement $E$ of $D$ would have to be an order ideal. Thus $E\wedge D=\{\emptyset\}$; hence by Theorem 7, $E\cap D=\{\emptyset\}$. Thus $E=\{\emptyset\}$. But this contradicts the fact that $E\vee D=B$, since $E\vee D=D\neq B$. □

*Remark.* One can define concepts analogous to continuity, $\gamma^*(T)$, and $S^\dagger$ using directed sets instead of chains. However, Example 3 shows that in general, one would get a different 'completion'. One can also consider the notion of '$\alpha$-completeness', i.e., posets in which every chain of cardinality not greater than $\alpha$ has a sup, and define concepts analogous to continuity, $\gamma^*(T)$, and $S^\dagger$ restricting the cardinality of allowable chains. We leave the details to the reader.

## 7. Fixpoints of chain-complete posets

The following result of Bourbaki, allows us to prove the basic fixpoint theorem for complete posets (Theorem 9) without using the axiom of choice. A proof of it may be found in [15, Theorem 1, p. 12].

THEOREM 8. *Let $P$ be a strictly inductive poset and $f:P\to P$ a map such that $x\leqslant f(x)$ for all $x\in P$. Then $f$ has a fixpoint.* □

THEOREM 9. *Let $P$ be a chain-complete poset, $f:P\to P$ isotone, and $F_P=\{x\in P \mid f(x)=x\}$ be the set of all fixpoints of $f$. Then:*

(i) *there exists a least element $0^*\in F_P$;*

(ii) *for all $y\in P$, if $f(y)\leqslant y$, $0^*\leqslant y$;*

(iii) *$F_P$ is a chain-complete in the induced order.*

*Proof.* (i) Let $S=\{x\in P \mid x\leqslant f(x) \text{ and } x\leqslant y \text{ for all } y\in F_P\}$. $0\in S$. $S$ is chain-complete. Let $C=\{x_\alpha\}_{\alpha\in\Delta}\subset S$ be a chain and $x$-$\sup_P C$. For all $\alpha\in\Delta$, $f(x)\geqslant f(x_\alpha)\geqslant x_\alpha$, whence $f(x)\geqslant x$. Since each element of $F_P$ is an upper bound for $C$, $x\in S$. Similarly, it is easy to see that $f(S)\subset S$. From Theorem 8, it follows that $F_P\cap S\neq\emptyset$. Thus $F_P$ has a least element $0^*$.

(ii) The set $[0,y]\ (=_{\text{def}}\{z\in P \mid 0\leqslant z\leqslant y\})$ is chain-complete, and $f([0,y])\subset[0,y]$. By (i), $F_P\cap[0,y]\neq\emptyset$. Thus $0^*\leqslant y$.

(iii) Let $C=\{x_\alpha\}_{\alpha\in\Delta}\subset F_P$ be a chain, and $u=\sup_P C$. $[u, -]$ $(=_{\mathrm{def}}\{z\in P \mid z\geqslant u\})$ is chain-complete. It is easily verified that $f([u, -])\subset[u, -]$. By (i) there is a least fixpoint $\gamma$ in $F_P\cap[u, -]$. Clearly, $\gamma=\sup_{F_P} C$.

COROLLARY. (Tarski) *If $P$ is a complete lattice, then so is $F_P$.*

*Proof.* Let $a$, $b\in F_P$ and $c=a\vee_P b$. As in (iii) $[c, -]$ is chain-complete and $f([c, -])\subset[c, -]$. By (i) there is a least fixpoint $\gamma\in F_P\cap[c, -]$. Clearly $\gamma=a\vee_{F_P} b$. $F_P$ is a complete lattice by (iii) and Corollary 5 of Theorem 1. □

*Remark.* Every poset with a least element satisfying the ascending chain condition is a chain-complete poset. Hence Theorem 9 generalizes the results in [9, p. 17].

Our next theorem generalizes Theorem 2 of [24] to chain-complete posets. When we speak of a commuting family of functions we mean that composition is commutative.

THEOREM 10. *Let $P$ be a chain-complete poset and $F$ a commuting family of isotone self-maps of $P$. Let $C$ be the set of common fixpoints of $F$, i.e., $C=\{x\in P \mid f(x)= =x$ for all $f\in F\}$.*
*Then:*

(i) *there exists a least element $0^*\in C$;*
(ii) *for all $y\in P$, if $f(y)\leqslant y$ for all $f\in F$, then $0^*\leqslant y$;*
(iii) *$C$ is chain-complete with respect to the induced order;*
(iv) *if $P$ is a lattice, then $C$ is a complete lattice* (Tarski).

*Proof.* We only prove (i), since the proofs of (ii)–(iv) can be modeled on the proofs used in Theorem 9.

Let $A=\{x\in P \mid f(x)\geqslant x$ for all $f\in F$ and $x\leqslant y$ for all $y\in C\}$. Clearly $0\in A$. If $g\in F$, $x\in A$, and $y\in C$, $g(x)\leqslant g(y)=y$. If in addition, $f\in F$, $f(g(x))=g(f(x))\geqslant g(x)$. Thus $g(A)\subset A$ for all $g\in F$. It is easy to see that $A$ is chain-complete. By Zorn's Lemma, $A$ has a maximal element $0^*$. But clearly $0^*\in C$ and it is the least element of $C$. □

*Remark.* We can avoid the use of Zorn's Lemma in the proof of Theorem 10 if we assume that $F$ is well-ordered, say $F=\{f_\lambda\}_{\lambda<\beta}$. Then we can define $f:A\to A$ by transfinite induction as follows. For $x\in A$, let $x_0=f_0(x)$. For $\lambda<\beta$, let $x_\lambda=f_\lambda(\sup_P \{x_\gamma\}_{\gamma<\lambda})$. Finally, let $f(x)=\sup_P\{x_\lambda\}_{\lambda<\beta}$. Clearly, $f(x)\geqslant x$ for all $x\in A$. By Theorem 8, $f$ has a fixpoint $a\in A$, i.e., $a=a_\lambda$ for all $\lambda<\beta$. Thus $a\in C$.

Davis [10] showed that a lattice is complete if and only if every isotone self-map has a fixpoint. Chain-complete posets cannot be characterized so easily. Take any poset $P$ in which every chain has an inf but which lacks a least element. The dual of Theorem 9 shows that every isotone self-map of $P$ has a fixpoint, but $P$ need not be complete. The next theorem characterizes chain-complete posets in terms of the existence of fixpoints. It also provides a partial answer to the question raised by Davis [10] as to whether a lattice $L$, having the property that every meet-preserving map $f:L\to L$ has a fixpoint, is necessarily complete. A meet-preserving map satisfies $f(a\wedge b)=f(a)\wedge f(b)$ whenever $a\wedge b$ exists in the domain of $f$. Inf-preserving maps are of course meet-preserving.



THEOREM 11. *Let $P$ be a poset. Then the following are equivalent.*

(a) *$P$ is chain-complete.*
(b) *Every isotone $f:P\to P$ has a* **least** *fixpoint.*
(c) *Every inf-preserving map $f:P\to P$ has a* **least** *fixpoint.*

*Proof.* Theorem 9 implies that (b) and (c) are consequences of (a). Clearly, (c) follows from (b). Thus we need only show that (c) implies (a).

Let $C\subset P$ be a chain. We have remarked above that every chain has a well-ordered cofinal subset, so that we may assume $C$ is well-ordered. Let $\mathscr{U}$ be the set of upper bounds of $C$ in $P$. Let $f:P\to P$ be given as follows: $f(x)=x$ if $x\in\mathscr{U}$; $f(x)=$least $y\in C$ such that $y\nleqslant x$ if $x\notin\mathscr{U}$.

To show that $f$ is inf-preserving, we let $X\subset P$ be such that $\inf X$ exists. If $X\subset\mathscr{U}$, $f(\inf X)=\inf X=\inf f(X)$, since $\inf X\in\mathscr{U}$. If $X\not\subset\mathscr{U}$, let $B=X-\mathscr{U}$. Thus, $\operatorname{Inf} X\notin\mathscr{U}$. But $\inf f(X)=\inf f(B)=$least $y\in C$ such that $y\in f(B)$ (call it $y_0$). Now $y_0\nleqslant\inf X$, since if $y_0\leqslant\inf X$, $y_0\leqslant x$ for all $x\in X$ and $y_0\notin f(B)$. For all $y\in C$, if $y<y_0$, $y\leqslant x$ for all $x$, i.e., $y\leqslant\inf X$. Thus $f(\inf X)=y_0=\inf f(X)$.

By hypothesis $f$ has a least fixpoint $\gamma\in P$. Every point of $\mathscr{U}$ is a fixpoint, and $\mathscr{U}$ is exactly the set of all fixpoints of $f$ since for $w\notin\mathscr{U}$, $f(w)\nleqslant w$. Thus $\gamma=\inf\mathscr{U}$, i.e., $\gamma=\sup C$. Hence $P$ is chain-complete. □

*Remark.* The proof of Theorem 11 can be modified to show that $P$ is a complete poset if and only if every map $f:P\to P$ of the form $f=g\circ h$ ($g$, $h:P\to P$, $g$ is sup-preserving, $h$ is inf-preserving) has a fixpoint. Dually, we can require $g$ to be inf-preserving and $h$ to be sup-preserving.

The author has used Theorem 9 to establish the existence of inverse limits in categories of complete posets (see [17]). Other applications of fixpoint theorems are in [7], [12], [19] and of course [24].

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