Semigroup Forum, Vol. 4 (1972), 341-344.

CARDINALITIES OF $\mathcal{D}$-CLASSES IN $\mathcal{B}_n$

R. L. Brandon, K. K.-H. Butler, D. W. Hardy, and G. Markowsky

Communicated by A. H. Clifford

Let $\mathcal{B}_n$ be the semigroup of all binary relations on a set of n elements. Let D be a $\mathcal{D}$-class of $\mathcal{B}_n$ with row rank s and column rank t and whose lattice has p elements. Then the number of elements in an L (R)-class of D is given by

$$\sum_{i=0}^{s} (-1)^i \binom{s}{i} (p - i)^n \qquad \left(\sum_{i=0}^{t} (-1)^i \binom{t}{i} (p - i)^n\right).$$

We find it convenient to phrase the argument in the terminology of matrix theory. We follow the notation introduced in [3] and [4] and use freely the results proved there; otherwise the notation and terminology is that of Clifford and Preston [6].

Let S be the semigroup $\mathcal{B}_n$ of all binary relations on a set of n elements, and let the elements of S be represented as $n \times n$ matrices over the Boolean algebra $\{0, 1\}$ of order 2. Such matrices will be called Boolean relation matrices. With each Boolean relation matrix A, there is associated a row (column) space $R(A)$ $(C(A))$ and it is well-known that two elements A and B of S are L (R) -equivalent iff $R(A) = R(B)$ $(C(A) = C(B))$ [4, 7]. For $A \in S$ the basis of $R(A)$ $(C(A))$ is called the row (column) basis of A and its cardinality is called the row (column) rank of A and denoted by $\rho_r(A)$ $(\rho_c(A))$. It is also known that every finite row (column) space has a unique "basis" [4]. Let $A \in S$. Then $B \in L_A$ (R_A) iff

$R(A) = R(B)$ $(C(A) = C(B))$ which implies that every element of the "basis" of $R(A)$ $(C(A))$ must appear at least once as a row (column) of B, while the remaining rows (columns) of B may be any elements of $R(A)$ $(C(A))$. Hence $|L_A|$ $(|R_A|)$ is the number of permutations of $|R(A)|$ $(|C(A)|)$ objects of which every one of $\rho_r(A)$ $(\rho_c(A))$ appears at least once. The following counting lemma is a key step in proving the main result.

LEMMA 1. *The number of sequences of length n of objects selected in all ways from p objects such that k specified objects must appear each time is*

$$\psi(p, k, n) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (p - i)^n,$$

where $\binom{k}{i}$ is the binomial coefficient.

Proof. We will prove this lemma by using generating functions. The generating function for the situation described above will be

$$(t + t^2/2! + t^3/3! + \ldots)^k (1 + t + t^2/2! + \ldots)^{p-k}$$

$$(*) \qquad = (e^t - 1)^k (e^t)^{p-k} = \left(\sum_{i=0}^{k} (-1)^i \binom{k}{i} e^{(k-i)t}\right) e^{(p-k)t}$$

$$= \sum_{i=0}^{k} (-1)^i \binom{k}{i} e^{(p-i)t}.$$

But $e^{(p-i)t} = \sum_{n=0}^{\infty} (p - i)^n (t^n/n!)$. Thus (*) becomes

$$\sum_{n=0}^{\infty} \left(\sum_{i=0}^{k} (-1)^i \binom{k}{i} (p - i)^n\right)(t^n/n!).$$

We may obtain $\psi(p, k, n)$ directly using the principle of inclusion and exclusion [5].

In order to obtain the main result, we need the following fact. If $A \in S$, then $|R(A)| = |C(A)|$ [2, 7]. Having assembled the necessary machinery, we are going to give the following counting result.

THEOREM 2. *Let $A \in S$. Let $s = \rho_r(A)$, $t = \rho_c(A)$, $h = |H_A|$, and $p = |R(A)| = |C(A)|$. Then*

$$\text{(i)} \quad |L_A| = \sum_{i=0}^{s} (-1)^i \binom{s}{i} (p - i)^n,$$

$$\text{(ii)} \quad |R_A| = \sum_{i=0}^{t} (-1)^i \binom{t}{i} (p - i)^n,$$

$$\text{(iii)} \quad |D_A| = (|L_A||R_A|)/h.$$

Proof. The proofs of (i), (ii), and (iii) follow immediately from Lemma 1 and all the preceding discussions.

COROLLARY 3. *If $s = t$, as is the case if D_A is regular, then*

$$|D_A| = (|L_A|)^2/h.$$

The above results can be formulated in terms of lattices [1].

We conclude this paper by giving an example for counting the values of p in $\psi(p, k, n)$. If

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix},$$

then clearly $\rho_r(A) = 4 = \rho_c(A)$, and $|H_A| = 2$. Let

$$J = \begin{bmatrix} 0&1&0&0&0&1&1&1&0&0&0&1&1&1&0&1 \\ 0&0&1&0&0&1&0&0&1&1&0&1&1&0&1&1 \\ 0&0&0&1&0&0&1&0&1&0&1&1&0&1&1&1 \\ 0&0&0&0&1&0&0&1&0&1&1&0&1&1&1&1 \end{bmatrix}$$

denote an 4×2^4 matrix in which column represents all possible column vectors for $C(A)$. Then

$$AJ = \begin{bmatrix} 0&1&0&0&0&1&1&1&0&0&0&1&1&1&0&1 \\ 0&0&1&0&0&1&0&0&1&1&0&1&1&0&1&1 \\ 0&1&0&1&0&1&1&1&1&0&1&1&1&1&1&1 \\ 0&1&0&0&1&1&1&1&0&1&1&1&1&1&1&1 \end{bmatrix}.$$

Clearly AJ contains 10 different column vectors and so $p = 10$.

Let J^t be the transpose of J. Then J^tA contains 10 different row vectors and so $p = 10$. Hence we get

$$|D_A| = (10^n - 4(9)^n + 6(8)^n - 4(7)^n + 6^n)^2/2.$$

REFERENCES

1. R. L. Brandon and D. W. Hardy, Cardinalities of $\mathcal{D}$-classes in $\mathcal{B}_n$, Technical Report, Colorado State University, Fort Collins, Colorado, 1972.

2. ____________, and G. Markowsky, The Schutzenber-ger group of an $\mathcal{H}$-class in the semigroup of binary relations, Semigroup Forum, to appear.

3. K. K.-H. Butler, On (0,1)-matrix semigroups, Semigroup Forum, 3 (1971), 74-79.

4. ____________, Binary relations, Recent Trends in Graph Theory, Lecture Notes in Math., No. 186 (Springer-Verlag, Berlin, Heidelberg, and New York, 1971), 25-47.

5. ____________, An identity in combinations, Kyungpook Math. J., 11 (1971), 197-198.

6. A. H. Clifford and G. B. Preston, The Algebraic Theory of Semigroups, Vol. 1, Amer. Math. Soc., Providence, R. I., 1961.

7. K. A. Zaretskii, The semigroup of binary relations, Mat. Sbornik, 61 (1963), 291-305 (in Russian).

Colorado State University
Fort Collins, Colorado 80521

Pembroke State University
Pembroke, N. C. 28372

Colorado State University
Fort Collins, Colorado 80521

Harvard University
Cambridge, Massachusetts 02138

Received January 21, 1972