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A MOTIVATION AND GENERALIZATION OF SCOTT'S NOTION OF A CONTINUOUS LATTICE

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ABSTRACT

In [6], Scott introduced continuous lattices as the correct setting for an abstract theory of computation. The motivation and definition of continuous lattices was primarily in topological terms. In [7], Scott discussed continuous lattices primarily from a topological point of view. However, buried in [7] is an indication (without proof) of how to approach continuous lattices from a purely order-theoretic perspective. This order-theoretic approach seems to have escaped the notice of most computer scientists. We develop this approach in the more general setting of chain-complete posets and offer some arguments in support of the thesis that "continuous posets" (chain-complete posets with a basis) are the proper setting for an abstract theory of computation. Our definition of basis also generalizes that used by Markowsky and Rosen [5]. Finally, we discuss a number of constructions which construct posets with a basis from posets with bases.

1. Introduction

Following Dana Scott's lead [6], a number of authors (Egli and Constable [3], Markowsky and Rosen [5], Vuillemin [8]) have considered the problem of defining a basis for certain classes of posets. In Scott's language, these posets with a basis would be called "continuous." The Markowsky-Rosen approach [5], based on an analogy with the definition of a basis for a vector space, includes the definitions of Egli and Constable [3] and Vuillemin [8] as special cases. It does not include Scott's definition [6,7] as a special case in the context of lattices.

In this paper, we will use the concept of relative compactness. This concept is more general than that of IN (denoted as $\prec$ by Scott) used by Scott [6,7]. Furthermore, while Scott uses topological concerns to motivate his definition of IN, the definition of relatively compact can be motivated in a very simple manner purely on the basis of order-theoretic concerns which arise naturally in the theory of computation. We note that relative compactness occurs in [7] following the proof of Theorem 2.11, but Scott uses the same symbol to denote it and IN. However, he does not derive its properties nor the relationship between these two different concepts. Scott now seems to prefer relative compactness (which he calls "way below" and denotes by $<<$) as the basic concept in defining continuous lattices.

Using relative compactness, a basis for a poset may be defined. At first glance this concept of a basis would appear to be more general than Scott's (as translated for posets). However, it will be shown that in the presence of a basis, the concepts of relative compactness and of IN agree. Thus the two notions of basis coincide and we arrive at the class of continuous posets while avoiding topological considerations comp-

Furthermore, this concept of basis includes that of Markowsky and Rosen [5] as a special case. Finally, some conditions under which posets with bases can be constructed starting with other posets with bases will be presented.

All posets shall be assumed to be *chain-complete posets* (*CPO's*), i.e., posets in which *every* chain has a sup. In particular, CPO's have least elements which are the sup's of the empty chain. It follows [4; Corollary 2] that in a CPO every directed subset has a sup. Furthermore, it follows [4; p.56; Corollary 5] that if a CPO happens to be a lattice, then it is a complete lattice.

There are two reasons for being primarily interested in sups of chains and directed sets. First, in most posets of interest in computer science, sups of chains (and hence of directed sets) exist naturally or correspond to natural objects which can easily be added to the original poset. Second, attempts to make arbitrary (even finite) sups exist often require the introduction of elements which have no natural meaning and exist solely to be the sups of certain subsets. The following example will illustrate these points.

Let A be some alphabet containing at least two characters. Let P be the set of all nonempty finite words made from A. For $w,w' \in P$, we say that $w \leq w'$ iff w is a prefix of w'. Clearly, this relation partially orders P. Note that P lacks a least element, which would correspond to the empty word and be the sup of the empty chain. Clearly, the empty word is a very natural object and most authors would have included it in the definition of P. Furthermore, a bit of reflection will show that the sup of infinite chains in P can be represented by infinite words which are also natural objects to work with (e.g., the decimal expansions of real numbers). Thus aaa... would represent the sup of $a \leq aa \leq aaa \leq \ldots$. For distinct $a,b \in A$, there is no natural word-like object which would represent the sup of {a,b}.

An additional property worth discussing is that of possessing *bounded join (sups))*, i.e., whenever any finite (arbitrary) nonempty subset which has an upper bound has a sup. Chain-complete posets with bounded joins have sups [5; p.142] for all bounded sets and look like collections of complete lattices which have been combined together. They occur quite naturally, e.g., the poset in the preceding example has bounded joins.

In most posets of interest to computer science, the partial ordering is loosely based on the amount of "information" contained in the objects. Thus in the example above, if our goal is the infinite words, the word aaa gives more "information" about an infinite word than does aa, since we now know the third letter. The preceding point is useful motivation for the next section and gives some ideaa of the importance of the "limit" objects. Usually the "finite" ("finitely computable") object consist of incomplete information and only the limit objects fully describe some state.

2. Relatively Compact Elements and Bases

A basis always involves the concept of generation and often that of independence. In this case, the requirement that a set, S, generate an element, x, translates into the requirement that S be directed and that x = sup S. The notion of independence is a difficult one to capture in this setting. Markowsky and Rosen [5] have developed an interpretation of it which we will discuss later. Here we will offer a slightly more general approach than is used there.

The chief reason for insisting that elements of a basis be independent is to guarantee that the representation of an element by basis elements is nonredundant, i.e., that every element in the representation is essential and cannot be gotten rid of. Furthermore, the existence of a basis should tie the poset together, so that some relations between elements can be inferred from relations between their representations by basis elements.

2.1 Definition Let P be a poset and $x,y \in P$.

(i) x is said to be *compact* if for every *nonempty* directed set $D \subseteq P$ with $\sup D \geq x$, there exists $d \in D$ with $d \geq x$.

(ii) x is said to be *relatively compact to y*, xRCy, if for every *nonempty* directed set $D \subseteq P$ with $\sup D \geq y$ there exists $d \in D$ with $d \geq x$.

The above definition of compactness agrees with that of Birkoff [1; p.186] and Crawley and Dilworth [p.13]. Note that a compact element is one which contains some essential piece of "information," i.e., the su of a directed set embodies the "information" in x iff some element of the directed set already embodies th information in x. If P is a CPO, one may define compactness by replacing directed sets in Definition 2.1 (by chains [5; Lemma 2.5].

If xRCy, then x contains some essential information about y, which cannot be suddenly produced by directed set. Thus if we have arrived at a point containing more information than y, we have done so o because we have incorporated all of the information in x.

The following lemma summarizes some of the properties of the binary relation RC. Its proof is v straight-forward and hence has been omitted.

2.2 Lemma Let P be a poset and $x,y, \in P$.

(i) xRCx iff x is compact.

(ii) xRCy implies $x \leq y$.

(III) xRCy and $y \leq z$ imply xRCz.

(iv) $x \leq y$ and yRCz imply xRCz.

(v) xRCy and yRCz imply xRCz.

(vi) If $z = \sup \{x_1,...,x_n\}$ with $x_i,...,x_n$RCy, then zRCy. □

2.3 Definition Let P be a poset and $x \in P$.

(i) A *nonempty* set $D \subseteq P$ consisting of elements *relatively compact* to x is a *local basis for x* if D is di and $\sup D = x$.

(ii) P has a *basis* if every element of P has a local basis. A subset B is a *basis of P* whenever for $x \in P$ some subset of B is a local basis for x and if every element of B is in some local basis. For $x \in P$, B_x to denote this local basis.

2.4 Proposition Let P be a poset and $y \in P$. Then y has a local basis iff D_y, defined as $\{x \in P \mid xRCy\}$, is a local basis for y. Note that D_y is closed below ($z \leq x$ and $x \in D_y$ imply $z \in D_y$) by Lemma 2.2.

Proof: Since efficiency is trivial, we first prove necessity. Let D by any directed set of elements relatively compact to y such that y = sup D. For all $x \in D_y$, there exists $d_x \in D$ such that $x \leq d_x$. Note also that $D \subseteq D_y$, whence it follows easily that D_y is directed. By Lemma 2.2, $x \in D_y$ implies that $x \leq y$. Thus y is an upper bound for D_y. Since y = sup D and $D \subseteq D_y$, it follows that y = sup D_y. Thus D_y is a local basis for y. □

Throughout this paper, D_y will denote the set of all elements relatively compact to y. Proposition 2.4 shows that for a poset P with a basis, the set $B_P = \{x \mid xRCy \text{ for some } y \in P\}$ is a basis and contains all other bases. Thus, one can think of it as the natural basis for P. The following theorem is the key result which enables us to relate our definition of basis to the generalization of Scott's topological definition.

2.5 Theorem Let P be a poset with a basis, $x,y \in P$ with xRCy and $D \subseteq P$ a nonempty directed set such that sup $D \geq y$. Then there exists $z \in D$, such that xRCz. In particular, RC *interpolates*, i.e., there exists $z \in D_y$ such that xRCzRCy.

Proof: For all $z \in D_z$, let $D_z = \{t \in P \mid tRCz\}$. By Proposition 2.4, D_z is directed and sup $D_z = z$. Let $D^* = \bigcup_{z \in D} D_z$. If $z_1, z_2 \in D$ and $z_1 \leq z_2$, $D_{z_1} \subseteq D_{z_2}$ by Lemma 2.2. Thus D^* is directed. Furthermore, sup D^* = sup $D \geq y$. Thus for some $t \in D^*$, $x \leq t$. But $t \in D_z$ for some $z \in D$, whence xRCz by Lemma 2.2. □

Note that all chains have a basis. Furthermore, let X and Y be sets and Pfun(X,Y) be the set of all partial functions from X into Y, where a partial function is an ordered pair (S,f) with $S \subseteq X$ and $f: S \to Y$ a function. Pfun(X,Y) is ordered by $(S,f) \leq (T,g)$ iff $S \subseteq T$ and $g \mid S = f$. It is easy to see that (S,f) RC (T,g) iff $(S,f) \leq (T,g)$ and S is finite. In particular, for finite S, (S,f) is compact. Pfun(X,Y) has a unique basis - the set of all compact elements. Note also that Pfun(X,Y) is a natural example of a continuous poset which is not a lattice, although it does possess bounded sups.

3. Other Notations of Basis

In this section, the definitions of basis as studied by Scott [6, 7] will be compared to those of Markowsky and Rosen [5]. We will redo Scott's topological approach in the more general context of posets, but ignore the questions of countability raised by Scott in [6].

3.1 Definition Let P be a poset, $S \subseteq P$ and $x,y \in P$.

(i) S is said to be *open* if it is *closed above* ($t \in S$, $w \in P$ and $t \leq w$ imply $w \in S$) and for every nonempty directed $D \subseteq P$, sup $D \in S$ implies $D \cap S$ is nonempty.

(ii) xINy if there exists an open set $U \subseteq P$ such that $y \in U$ and $x \leq t$ for all $t \in U$.

(iii) A nonempty directed set $D \subseteq P$ such that tINy for all $t \in D$, is a *local Scott basis* for y if sup D=y.

(iv) P has a *Scott basis* if every point has a local Scott basis. A subset, B, of P is a *Scott basis of P* if for every element of P some subset of B is a local Scott basis and every element of B is in some local Scott basis.

3.2 Lemma Let P be a poset, $x,y \in P$. If xINy, then xRCy.

Proof: Let D be directed such that sup $D \geq y$. Since xINy, there exists an open set $U \subseteq P$ such that $y \in U$ and $x \leq t$ for all $t \in U$. Since U is open, sup $D \in U$, whence $d \in U$ for some $d \in D$. But this implies $d \geq x$, thus, xRCy. □

3.3 Example This example shows that xRCy can hold without xINy holding. Let $N^{\infty} = \{0,1,2,\ldots\} \cup \{\infty\}$ be ordered in the obvious way. Let $P = \{0\} \cup (\{-1\} \times N^{\infty})\ (N \times N^{\infty}) \cup \{(\infty,\infty)\}$ where N denotes the natural numbers. We order P as follows: 0 is the least element of P; $(m,n) \leq (m',n')$ with $m,m',n,n' \in N$ iff $m = m'$ and $n \leq n'$ in N; (∞,∞) is the greatest element of P; for $m \in N$, $(m,\infty) \geq (-1,n)$, (m',n') providing $m \geq n$, $m' \in N$; $(-1,m) \geq (-1,m')$ providing $m \geq m'$ in N^{∞}. The reader can verify that the above define a partial ordering on P.

Note that $(-1,0)RC(-1\infty)$ since the only nontrivial directed sets whose sups are $\geq (-1,\infty)$ are ones which include cofinal sections either of the chain $\{(-1,n) \mid n \in N\}$ or $\{(m,\infty) \mid m \in N\}$. We claim that $(-1,0)IN(-1,\infty)$ does not hold.

Let U be any open set containing $(-1,\infty)$. Since sup $\{(-1,\infty) \mid n \in N\} \in U$, $(-1,n) \in U$ for some $n \in N$. Since U is closed above, $(n,\infty) \in U$ for some $n \in N$. Since $(n,\infty) = \sup \{(n,m) \mid m \in N\}$, $(n,m) \in U$ for some $m \in N$. But $(n,m) \not\geq (-1,0)$. Thus no open set containing $(-1,\infty)$ consists entirely of elements $\geq (-1,0)$. Thus $(-1,0)IN(-1,\infty)$ fails to hold. □

3.4 Theorem Let P be a poset with a basis and $x,y \in P$. Then xRCy iff xINy. Thus any basis is a Scott basis and vice versa. Actually, the following proof shows that if x has the property that for all y, xRCy implies that there exists z with xRCzRCy, then xRCy implies xINy.

Pf: That xINy imples xRCy follows from Lemma 3.2. Now suppose xRCy. Let $U = \{z \in P \mid xRCz\}$. By Lemma 2.2, U is closed above and $x \leq z$ for all $z \in U$. By Theorem 2.5, if sup $D \in U$, $D \cap U$ is nonempty. Thus U is open and xINy. □

Following Scott's lead one can use the term *continuous poset* to denote any chain-complete poset which has a basis. In the context of lattices, Scott [6] required that finite sups of basis elements be basis elements. For lattices with a basis as in Definition 2.1 this requirement is met as a consequence of Lemma 2.2 (vi) and Proposition 2.4.

Note that Scott's continuous lattices are not continuous lattices in the sense of Crawley and Dilworth [2; p.15] although they must of necessity be "upper continuous" in the terminology of Crawley and Dilworth. Furthermore, Jeffrey Leon's example [2; p.16] shows that a lattice can be continuous in the sense of Crawley and Dilworth without possessing a basis.

We now turn our attention to the concept of basis studied by Markowsky and Rosen [5]. A basis, B, of a vector space, V, is a subset of V such that an arbitrary map $f:B \to W$ with W a vector space extends to a *unique* linear map $f^*:V \to W$. With this in mind, Markowsky and Rosen defined a basis, B, of a chain-complete poset, P, to be a subset such that an arbitrary *isotone* (order-preserving) map $f:B \to Q$ with Q a chain-complete poset extends to a *unique continuous* map (one preserving sups of *nonempty* directed sets; see [4;

Corollary 2] for more details). This definition does not lend itself to ready comparison with the other definitions presented here, although it makes clear how a basis is to satisfy the generation and independence conditions. However, Theorem 3.3 of [5] provides the necessary bridge to relate this definition of basis to the others. It asserts that a set B has the above mapping property iff it is in the set of compact elements of P and a local MR basis (see Definition 3.5 below) can be selected from B for each point of P.

3.5 Definition Let P be a poset and $y \in P$.

(i) A *nonempty* directed subset, D, of P consisting of compact elements is a *local MR basis* for y if sup $D = y$.

(ii) P has an *MR basis* if every point has local MR basis. A subset B of P is an *MR basis for P* if for every $x \in P$ some subset of B is a local MR basis for x and every element of B is in some local MR basis. A chain-complete poset with an MR basis can also be called *algebraic* (see [1; p.187]).

The following proposition is proved in the same way that Proposition 2.4 is proved. Hence we have omitted the proof.

3.5 Proposition Let P be a poset and $y \in P$. Then y has a local MR basis iff MR_y defined to be $\{x \in P \mid x$ is compact and $x \leq y\}$ is a local MR basis for y. Note MR_y need not be closed below. □

An MR basis arises naturally if one wants a basis so that each local basis is simply given by looking at the elements of some fixed set which are $\leq$ the element we are working with. Note that in our previous definition of basis we impose a basis locally and do not require much in the way of interrelation among the local bases. In view of Proposition 2.4 and Lemma 2.2 we can require that $x \leq y$ imply $D_x \subseteq D_y$. In the case of an MR basis, note that $x \leq y$ implies $MR_x = MR_y \cap (x]$, where $(x] = \{z \in P \mid z \leq x\}$. Theorem 3.7 shows that this regularity condition forces a basis to become an MR basis.

3.7 Theorem Let P be a poset. P has an MR basis iff there exists $B \subseteq P$ such that for all $y \in P$, $B \cap (y]$ is a local basis for y.

Proof: *Necessity* Let B be the set of all compact elements of P. It follows that $B \cap (y] = MR_y$, and by Proposition 3.6 this gives a local basis for y.

Sufficiency Let $x \in B$. Since $x \in B \cap (x]$, xRCx and by Lemma 2.2, x is compact. Thus B consists entirely of compact elements and thus $B \cap (y]$ is actually a local MR basis for each $y \in P$. Thus P has an MR basis. □

Furthermore, if P has an MR basis, B, then B is unique and must be the set of all the compact elements of P. Note that if P has a basis (Definition 2.1), B, then B must contain the set of all compact elements of P.

4. Constructions Using Posets With a Basis

In the section, we will discuss a number of poset constructions which produce posets with a basis starting from posets with a basis. The first construction we consider is the Cartesian product of posets ordered componentwise.

4.1 Theorem Let $\{P_\alpha\}_{\alpha\epsilon\Delta}$ be a family of posets, each with a least element and a basis, and $P=\Pi_{\alpha\epsilon\Delta}P_\alpha$ their Cartesian product ordered componentwise. Then P has a least element and a basis.

Proof: For each $a=(a_\alpha)\epsilon P$, let $D_a=\{(b_\alpha) \mid b_\alpha \epsilon P_\alpha$ such that $b_\alpha \neq 0$ for only finitely many α's and for those α's $b_\alpha RCa_\alpha\}$. The reader can verify easily that D_a is a local basis for a. Thus P has a basis. □

It is important to note that Theorem 4.1 could be false if the posets have a basis but lack least elements. This is illustrated in Example 4.2.

4.2 Example Let $P = \prod_{i=1}^{\infty} \mathbb{R}$. Given $a=(a_i)\leq(b_i)=b$, let $c_n=(c_{ni})$ be given by $c_{ni}=b_i$ for $i\leq n$ and $c_{ni}=a_i-1$ for $i>n$. Then $\{c_n\}$ is a chain with sup $\{c_n\}=b$. But for all n, $c_n \not\geq a_n$. Thus for all $a,b\epsilon P$, aRCb *fails* to hold. Thus P lacks a basis. □

The reader can verify that finite Cartesian products of posets with basis have a basis.

4.3 Definition Let P and Q be posets.

(i) A map $f:P\rightarrow Q$ is said to be *continuous* if for every *nonempty* directed set $D\subseteq P$ such that sup D exists in P, f(sup D) is the sup of f(D).

(ii) Let Con(P,Q) denote the poset of all continuous maps from P into Q. Here $f\leq g$ iff for all $x\epsilon P$, $f(x)\leq g(x)$. Note that Scott uses $[P\rightarrow Q]$ to denote this poset.

Note that continuous maps are isotone. Furthermore, if P is chain-complete a map is continuous iff for every nonempty chain $C\subseteq P$, f(sup C)=sup f(C) [4; p.56, Corollary 3].

The poset Con(P,Q) is used in a number of constructions and it is of interest to know when Con(P,Q) has a basis given that P and Q each have a basis. The following example shows that it is possible for P and Q to be a chain-complete and have bases without Con (P,Q) having a basis. (Note that if Q is chain-complete so is Con (P,Q).) This example is essentially Example 4.2 from [5].

4.4 Example Let $P=\{0,a,b,c_0,c_1,...\}$ be ordered by: 0 is the least element of P; a and b are $<c_i$ for all i; $c_i<c_j$ for i>j. Every element of P is compact and hence P has a basis. However, Con (P,P) does not have a basis. In particular, we will show that the identity map f: $P\rightarrow P$ does not have a local basis in Con (P,P).

The reader can easily verify that the maps $g:P\rightarrow P$ given by $g(x)=a$ if $x\geq a$, $g(0)=g(b)=0$ and $h:P\rightarrow P$ given by $h(x)=b$ if $x\geq b$, $h(0)=h(a)=0$ are continuous and relatively compact to f. If there were local basis for f, we could find a continuous map $f^*:P\rightarrow P$ such that f*RCf and $g,h\leq f^*$. If $f\geq f^*\geq g,h$, $f^*(\{c_0,c_1,...\})$ is an infinite subset of $\{c_0,c_1,...\}$.

Let $f^*_n(x)$ be given by x for $x\geq c_n$, x=a,b or 0, and otherwise by c_{i+1} where $f^*(x)=c_i$. Clearly $\sup\{f^*_n(x)\}=f(x)$ but $f^*\not\leq f^*_n$ for all n. This contradicts the fact that f* RCf. □

In general, if P and Q both have bases, the reason Con(P,Q) lacks a basis is that we cannot generally find *directed* subsets of relatively compact elements with the desired sups. The following theorem clarifies these matters somewhat. These results are generalizations of Scott's Theorem 3.3 [7].

4.5 Theorem Let P and Q be posets with basis, and assume that Q has a least element 0. For $p \in P$ and $q \in Q$, let $f(p,q): P \to Q$ be given by $f(p,q)(x) = q$ if pRCx and $f(p,q)(x) = 0$ otherwise.

(i) For all $p \in P$, $q \in Q$, $f(p,q) \in Con(p,q)$.

(ii) For all $p \in P$, $q \in Q$ and $g \in Con(P,Q)$, qRCg(p) implies that f(p,q)RCg. Furthermore, if $g \leq \sup D$ for some nonempty directed subset $D \subseteq Con(P,Q)$, then f(p,q)RCh for some $h \in D$.

(iii) For all $g \in Con(P,Q)$, $g = \sup \{f(p,q) \mid p \in P, q \in Q \text{ and } qRCg(p)\}$.

Proof:

(i) Let $D \subseteq P$ be a nonempty directed set with $\sup D = x$. If pRCx fails to hold, $f(p,q)(x) = 0$. Since for all $d \in D$, $d \leq x$, pRCd fails to hold for all d, $\sup (f(p,q)(D)) = 0$. If pRCx, then by Theorem 2.5, pRCd^* for some $d^* \in D$. Thus $q = f(p,q)(d^*) = f(p,q)(x)$, whence $f(p,q)(x) = \sup f(p,q)(D)$.

(ii) Suppose $\{g_a\}_{a \in D}$ is a directed subset of Con(P,Q) such that $\sup \{g_a\}$ exists in Con(P,Q) and is $\geq g$. In particular, $\sup \{g_a(p)\} \geq g(p)$. Since qRCg(p), by Theorem 2.5 it follows that qRC$g_{a^*}(p)$ for some $a^* \in D$. We now claim that $f(p,q)(x) \leq g_{a^*}(x)$ for all $x \in P$. We need only consider the case with pRCx, else $f(p,q)(x) = 0$. If pRCx, $p \leq x$, whence $g_{a^*}(x) \geq g_{a^*}(p) \geq q = f(p,q)(x)$. Thus $f(p,q) \leq g_{a^*}$. Note that the above proof repeated actually yields f(p,q)RCg_{a^*}.

(iii) Since P has a basis, for all x, $x = \sup D_x$ where D_x is the directed set of all p such that pRCx. Since g is continuous $g(x) = \sup \{g(p) \mid pRCx\}$. Since Q has a basis, $g(p) = \sup D_{g(p)}$ where $D_{g(p)}$ is the directed set of all $q \in Q$ such that qRCg(p). Thus $g(x) = \sup \{q \mid qRCg(p) \text{ and } pRCx\} = \sup \{f(p,q)(x) \mid qRCg(p) \text{ and } pRCx\} = \sup \{f(p,q)(x) \mid qRCg(p)\}$. Thus g is the pointwise sup of the f(p,q)'s, whence it must be their sup. □

Note that even though the set $\{f(p,q) \mid qRCg(p)\}$ need not be directed, for each $x \in P$, the set $\{f(p,q)(x) \mid qRCg(p)\}$ is directed. Thus we almost have a basis. If we wish to have a basis for Con(P,Q) we must impose further conditions on P and Q. The following theorem is a generalization of Theorem 4.5 of [5].

4.6 Theorem Let P and Q be as in Theorem 4.5. Further, assume that *either* P *or* Q has bounded joins. Then Con(P,Q) has a basis.

Proof:

(A) If Q has bounded joins, one easily verifies that Con(P,Q) does also. Thus every finite subset of $\{f(p,q) \mid qRCg(p)\}$ has a sup and the collection of all these sups is a directed set whose limit is g by Theorem 4.5. Furthermore, by Theorem 4.5 and Lemma 2.2 each of these sups is relatively compact to g. Thus for each g we have constructed a local basis, whence Con(P,Q) has a basis.

(B) If P has bounded joins, we will show that for each $g \in Con(P,Q)$ the set $D'_g = \{\sup A \mid A$ a nonempty finite subset of $\{f(p,q) \mid qRCg(p)\}$, which has a sup in $Con(P,Q)\}$ is a local basis or g. Clearly, by Theorem 4.5 $\sup D'_g = g$ and by Theorem 4.5 and Lemma 2.2 every element of D'_g is relatively compact to g. It remains only to show that D'_g is directed.

Suppose $g_1, g_2 \in D'_g$, we must find $g_3 \in D'_g$ with $g_1, g_2 \le g_3$. Suppose $g_1 = \sup A_1$ and $g_2 = \sup A_2$ with A_1 and A_2 finite subsets of the type described earlier.

Let $M = \{p \in P \mid$ for some $q \in Q$, $f(p,q) \in A_1 \cup A_2\}$. M is a nonempty, finite set. Let $N = \{p^* \in P \mid p^* = \sup F$ for some $F \subseteq M\}$. Note that each subset F of M either has a sup or has no upper bounds at all. List the members of N as $p^*_1, \ldots, p^*_k$ where $p^*_i < p^*_j$ implies $i<j$. For $i=1,\ldots,k$, let $F_i = \{p \in M \mid p \le p^*_i\}$. Clearly, F_i is the largest subset of M whose sup equals p^*_i.

We define a sequence $q_1, \ldots, q_i \in Q$ by induction so that it has the following properties: (a) $q_i RCg(p^*_i)$ for $i=1,\ldots,k$; (b) if $p^*_i \le p^*_j$, then $q_i \le q_j$; (c) if $p^*_i \in M$, $q_i \ge q$ for all $q \in Q$ such that $f(p^*_i,q) \in A_i \cup A_2$. Assume that we have defined q_i for $i<m$. Let $D_{g(p_i^*)}$ be the local basis for $g(p^*_i)$. Let $H = \{q_j \mid p^*_j < p^*_m\} \cup \{q \mid f(p^*_m,q) \in A_1 \cup A_2\}$. H is easily seen to be a finite subset of $D_{g(p^*_m)}$. Thus we can find $q_m \in D_{g(p^*_m)}$ which is an upper bound for H. Clearly, q_m satisfies (a), (b) and (c).

We note that the q_i have the following property: for $x \in P$, if $p^*_i RCx$ but $p^*_m RCx$ fails for $m>i$, then $q_i \ge q_j$ for all j such that $p^*_j RCx$. To see this let $M_x = \{p \in M \mid pRCx\}$. Since M_x is nonempty and is bounded above by x, $\sup M_x = p^*_t$ for some t. Note that $F_i \subseteq M_x$, whence $p^*_t \ge p^*_i$. Since $p^*_m RCx$ fails for all $m>i$, and $t \ge i$, $t=i$ and $M_x = F_i$. If $p^*_j RCx$, then $F_j \subseteq M_x = F_i$, whence $p^*_j \le p^*_i$ and $q_j \le q_i$ by (b) above.

Let g_3 be given by

$$g_3(x) = q_i \text{ if } p^*_i RCx \text{ by } p^*_m RCx \text{ fails for all } m>i$$

$$= 0 \text{ if } p^*_i RCx \text{ fails for all } i=1,\ldots,k.$$

From the preceding paragraph it follows that g_3 is isotone. Arguing as in the proof of Theorem 4.5(i) shows that g_3 is continuous. It is now easy to see that g_3 is the sup of $\{f(p^*_i,q_i) \mid i=1,\ldots,k\}$ in $Con(P,Q)$.

Since $q_i RCg(p^*_i)$, by Theorem 4.5 and Lemma 2.2 we get that $g_3 RCg$. By (c) above it follows that $g_3 \ge f(p,q)$ for all $f(p,q) \in A_1 \cup A_2$, whence $g_3 \ge g_1, g_2$. □

We will not discuss the generalization of the material on recursive listings for an MR basis of a poset which is in [5]. Matters become a good deal more complicated and should be dealt with separately.

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