# ENUMERATION OF FINITE TOPOLOGIES

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Let X be a set with n elements. The number of topologies that can be defined on X has been determined for certain small values of n (see [4], [6]-[9], [12],[14], [17], and [18]). Also, the number of topologies defined on n points has been estimated by several authors (see [7], [8], and [11]).

In this paper, we shall present some formulas relating: (i) the number of $T_0$-topologies on n points with the number of topologies on n points; (ii) the number of connected $T_0$-topologies on n points with the number of connected topologies on n points; (iii) the number of isomorphism classes of $T_0$-topologies on n points with the number of isomorphism classes of connected $T_0$-topologies on n points; (iv) the number of isomorphism classes of topologies on n points with the number of isomorphism classes of connected topologies

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on n points; (v) the number of $T_0$-topologies on n points with the number of connected $T_0$-topologies on n points; and (vi) the number of topologies on n points with the number of connected topologies on n points. We accomplish this by considering an equivalent problem. Namely, we shall use the correspondence between $T_0$-topologies and posets on n points and the correspondence between topologies and quasi-ordered sets on n points, respectively (see [1] and [2]). The reader is refer to [2] and [13] for notation and terminology concerning ordered sets, and combinatorics, respectively.

Let $Q(X)$ be the set of all quasi-orders that can be defined on $X$; let $P(X)$ be the set of all partial orders that can be defined on $X$. Let $Q$ be an element of $Q(X)$ and let $x, y \in Q$. We define $x \equiv y$ iff $x \leq y$ and $y \leq x$.

LEMMA 1. $\equiv$ is an equivalence relation.

Let $Q'$ denote the *quotient-set* $Q/\equiv$.

LEMMA 2. Let $Q$ be an element of $Q(X)$. Then $Q'$ is a partially ordered set in the obvious way: Let $x^*, y^* \in Q'$, then $x^* \geq y^*$ iff for some $x \in x^*$, $y \in y^*$, we have , $x \geq y$.



Proof. (i) $x^* \geq x^*$ for all $x^* \in Q'$.

(ii) $x^* \geq y^*$, $y^* \geq x^*$ for some $x^*, y^* \in Q$ => there exist $x, z \in x^*$, $y, w \in y^*$ such that $x \geq y$, $w \geq z$. But $x, z \in x^*$ => $z \geq x$ and $y, w \in y^*$ => $y \geq w$ => $y \geq w \geq z \geq x$ => $y \geq x$ (since $Q \in Q(X)$) => $x \in y^*$ => $x^* = y^*$ (since equivalence classes are equal if they have a common member).

(iii) $x^* \geq y^*$, $y^* \geq z^*$ => there exist $x \in x^*$, $y \in y^*$, $z \in z^*$ such that $x \geq y$, $y \geq z$. Thus $x \geq z$ => $x^* = z^*$.

A quasi-ordered set $Q$ is said to be *connected* if for any $x, y \in Q$ there exists a sequence $q_1, \ldots, q_k$ in $Q$ such that $x = q_1$, $y = q_k$ and for each $i = 1, \ldots, k - 1$ either $q_i \leq q_{i+1}$ or $q_i \geq q_{i+1}$. Let $Q^c(X)$ be the set of all connected quasi-orders that can be defined on $X$; let $P^c(X)$ be the set of all connected partial orders that can be defined on $X$.

LEMMA 3. Let $Q \in Q(X)$. Then $Q \in Q^c(X)$ iff $Q' \in P^c(X)$.

Let $s(n, m)$ and $S(n, m)$ denote the Stirling numbers of the first and second kinds respectively (see [13]). Let $P$ be a partial order defined on $m$ distinct elements, $a_1, \ldots, a_m$; and let $I(P)$ be the number of partial orders defined on $a_1, \ldots, a_m$ which are isomorphic to $P$.

LEMMA 4. (i) Let $P$ be a partially ordered set defined on $m$ distinct elements, $a_1, \ldots, a_m$. Then there exist $S(n, m)I(P)$ quasi-ordered sets $Q$ defined on the $n$ distinct elements, $b_1, \ldots, b_n$ such that $Q'$ is isomorphic to $P$ (in symbols, $Q' \simeq P$).

(ii) Let $P$ be a connected partially ordered set defined on $m$ distinct elements, $a_1, \ldots, a_m$. Then there exist exactly $S(n, m)I(P)$ connected quasi-ordered sets $Q$ defined on the $n$ distinct elements, $b_1, \ldots, b_n$ such that $Q'$ is isomorphic to $P$.

Proof. (i) $Q' \simeq P$ implies that the $b_i$ are partitioned into $m$ nonempty disjoint equivalence classes, and that these equivalence classes are ordered in some way isomorphic to $P$. Thus for any partition of the $b_i$ into $m$ such equivalence classes there are $I(P)$ ways of ordering these equivalence classes in a way isomorphic to $P$. But [13, p. 99] it is shown that there are $S(n, m)$ distinct ways to partition $n$ elements into $m$ nonempty disjoint sets. Thus the number of suitable quasi-ordered sets is $S(n, m)I(P)$. This result holds for any partially ordered set $P$.

(ii) From Lemma 3 it follows that if $P$ is a connected partially ordered set there exist $S(n, m)I(P)$ connected quasi-ordered sets $Q$ such that $Q' \simeq P$.



Let $Q(n)$ be the number of quasi-orders on $X$; let $Q^c(n)$ be the number of connected quasi-orders on $X$; let $P(n)$ be the number of partial orders on $X$; and let $P^c(n)$ ~~be~~ the number of connected partial orders on $X$. Let $T(X)$ be the set of all topologies that can be defined on $X$; let $T^c(X)$ be the set of all connected topologies that can be defined on $X$; let $T_o(X)$ be the set of all $T_o$-topologies that can be defined on $X$; and let $T_o^c(X)$ be the set of all connected $T_o$-topologies that can be defined on $X$. Let $|Y|$ denote the cardinality of a set $Y$. Let $T(n) = |T(X)|$, $T^c(n) = |T^c(X)|$, $T_o(n) = |T_o(X)|$, and $T_o^c(n) = |T_o^c(X)|$ respectively.

THEOREM 5. (i) $Q(n) = \sum_{m=1}^{n} S(n, m)P(m)$,

(ii) $Q^c(n) = \sum_{m=1}^{n} S(n, m)P^c(m)$.

Proof. (i) For each quasi-ordered set $Q$ on $n$ elements $Q'$ is isomorphic to a partially ordered set $P$ on $m$ elements, $m \leq n$.

(ii) For each connected quasi-ordered set $Q$ on $n$ elements $Q'$ is isomorphic to a connected partially ordered set $P$ on $m$ elements, $m \leq n$. The theorem now follows from Lemma 4.

REMARK 1. Theorem 5 (i) was proved by Evans, Harary, and Lynn [9] in a similar manner but using graph-theoretic techniques.

COROLLARY 6. (i) $P(n) = \sum_{m=1}^{n} s(n, m)Q(m)$,

(ii) $P^c(n) = \sum_{m=1}^{n} s(n, m)Q^c(m)$.

Proof. Follows from Stirling number inversion [13, p. 34].

Let $Q$ be an element of $Q(X)$. We define the relation $\xi$ as follows: If $x, y \in Q$, then $x \xi y$ iff there exist $a_1, \ldots, a_k \in Q$ such that $x = a_1$, $y = a_k$ and for $i = 2, \ldots, k$ either $a_{i-1} \leq a_i$ or $a_{i-1} \geq a_i$. Clearly $\xi$ is an equivalence relation. The equivalence classes generated by $\xi$ are called the *connected components* of $Q$.

REMARK 2. The connected components of $Q$ are themselves connected quasi-ordered sets if considered separately. If $Q$ is a partially ordered set, then the connected components are also partially ordered sets if considered separately.

Let $\bar{Q}(n)$, $\bar{Q}^c(n)$, $\bar{P}(n)$, and $\bar{P}^c(n)$ be the number of isomorphism classes in $Q(X)$, $Q^c(X)$, $P(X)$, and $P^c(X)$ respectively.



Let $\bar{T}(n)$, $\bar{T}^c(n)$, $\bar{T}_o(n)$, and $\bar{T}^c(n)$ be the number of isomorphism classes in $T(X)$, $T^c(X)$, $T_o(X)$, and $T_o^c(n)$ respectively.

By $\pi_n$ we mean the set of all unordered partitions of $n$. If $k \in \pi_n$, by $k_i$ we mean the number of times $i$ appears as a part of $k$. Thus if $k = 1, 2, 2$ is a partition of 5, $k_1 = 1$, $k_2 = 2$, $k_3 = k_4 = k_5 = 0$. We define $k(\pi) = \sum_{i=1}^{n} k_i$. For the sake of simplicity, we let

$$\Phi_i = \binom{\bar{Q}^c(i) + k_i - 1}{k_i} \quad \text{and} \quad \Psi_i = \binom{\bar{P}^c(n) + k_i - 1}{k_i}.$$

THEOREM 7. (i) $\bar{P}(n) = \sum_{\pi_n} \left( \prod_{i=1}^{n} \Psi_i \right)$,

(ii) $\bar{Q}(n) = \sum_{\pi} \left( \prod_{i=1}^{n} \Phi_i \right)$.

Proof. We will only prove (i) since the proof of (ii) is almost identical. Each partially ordered set of $n$ elements induces a partition of $n$, simply by considering the cardinalities of the connected components of the given partially ordered set. Generally, this is a many-to-one correspondence. Clearly, two partially ordered sets are isomorphic iff they both have the same number of connected components and the connected components are of these two partially ordered sets are isomorphic in some order.

Suppose we are given a partition $k$ of $n$. We wish to show that there are exactly

$$\prod_{i=1}^{n} \binom{\bar{P}^c(i) + k_i - 1}{k_i}$$

isomorphism classes of partially ordered sets whose connected components induce $k$. $\Psi_i$ is the number of combinations (repetitions allowed) of $k_i$ objects chosen from $\bar{P}^c(i)$ objects, i. e., it is exactly the number of different ways in which it is possible to put a partial order structure on $k_i$ sets of $i$ elements each. Since partially ordered sets with different numbers of elements can not be isomorphic, it follows that the choices of partial order structure for the components of size $i$ are mutually independent and hence there are exactly

$$\prod_{i=1}^{n} \Psi_i$$

isomorphism classes of partially ordered sets whose connected components induce $k$. The theorem now follows immediately.



REMARK 3. Put $\rho_n = \bar{P}(n)$, $g_n = \bar{P}^c(n)$, $\rho(x) = \rho_o + \rho_1 x + \rho_2 x^2 + \ldots + \rho_n x^n + \ldots$ $(\rho_o = 1)$, $g(x) = g_1 x + g_2 x^2 + \ldots + g_n x^n + \ldots$ . The relations in Theorem 7 imply [13]

$$\rho(x) = (1 - x)^{-g_1}(1 - x^2)^{-g_2} \ldots (1 - x^n)^{-g_n} \ldots$$

$$= \exp\,(g(x) + g(x^2)/2 + \ldots + g(x^n)/n + \ldots)$$

$$= \exp \sum_1 g_n^* x^n/n,$$

where $g_n^* = \sum_{d|n} d g_d$. Differentiation of the last leads to

$$\rho_n = (g_n^* + \rho_1 g_{n-1}^* + \ldots + \rho_{n-1} g_1^*)/n.$$

The Bell polynomial is defined by

$$Y_n(fy_1, fy_2, \ldots, fy_n) = \sum_{\pi_n} \frac{nf_k}{k_1!k_2! \ldots k_n!} \left(\frac{y_1}{1!}\right)^{k_1} \left(\frac{y_2}{2!}\right)^{k_2} \ldots \left(\frac{y_n}{n!}\right)^{k_n},$$

where $\pi_n$ denotes a partition of $n$, usually denoted by $1^{k_1}2^{k_2} \ldots n^{k_n}$, with $k_1 + 2k_2 + \ldots + nk_n = n$; $k_i$ is, of course, the number of parts of size $i$. Also $f^k \equiv f_k = (-1)^{k-1}(k - 1)!$ [13]. For basic properties of the $Y_n$, the reader is referred to (see [13], Sections 2.8 and 4.5). Let $Y_n(y_1, y_2, \ldots, y_n)$ denote the Bell polynomial with all $f_i$ set at unity. This particular Bell polynomial may be interpreted as an ordered-cycle indicator [13, p. 75].

THEOREM 8.

(i) $P(n) = Y_n(P^c(1), P^c(2), \ldots, P^c(n))$,

(ii) $P^c(n) = Y_n(fP(1), fP(2), \ldots, fP(n))$, $f_k = (-1)^{k-1}(k-1)! = s(k, 1)$,

(iii) $Q(n) = Y_n(Q^c(1), Q^c(2), \ldots, Q^c(n))$,

(iv) $Q^c(n) = Y_n(fQ(1), fQ(2), \ldots, fQ(n))$, $f_k = (-1)^{k-1}(k-1)! = s(k, 1)$.

Proof. We will only prove (i) since the proof of (iii) is similar, and (ii) and (iv) are just the inverses of (i) and (iii) respectively. As in the proof of Theorem 7 we note that each partially ordered set of $n$ elements induces a partition of $n$ simply by considering the cardinalities of the connected components of the given partially ordered set. It is easy to see that there are

$$\frac{n!}{k_1(1!)^{k_1} k_2(2!)^{k_2} \ldots k_n(n!)^{k_n}}$$

distinct ways (up to isomorphism) of distributing $n$ distinct elements into $k(\pi)$ parts (where there are $k_i$ parts of size $i$). On each of these $k(\pi)$ parts we are free to set up any partial ordering we wish and the resulting partial orderings on $X$ will all be distinct since different groups of distinct elements are involved. Thus the theorem follows immediately.



REMARK 4. Write $\rho_{nm}$ for the number of orderings of $n$ elements with $m$ labeled, $g_{nm}$ for the similar variable. Then, if

$$g(x, y) = \sum_{\substack{n=1 \\ m=1(1)n}} g_{nm} x^n y^m / m!\ ,$$

$$\rho(x, y) = \sum_0 \rho_{nm} x^n y^m / m!$$

$$= \exp\,(g(x, y) + g(x^2)/2 + \ldots + g(x^n)/n + \ldots).$$

$$\rho(x, y) \exp g(x) = \rho(x) \exp g(x, y),$$

$$(\rho_x(x, y) + \rho(x, y) g'(x)) \rho(x) = (\rho'(x) + \rho(x) g_x(x, y)) \rho(x, y)$$

$$\rho_y(x, y) = g_y(x, y) \rho(x, y).$$

Employing the techniques in [13, p. 135] to determine the fully labeled case with $P_n = \rho_{nn}$, $R_n = g_{nn} = P^c(n)$,

$$\sum_0 P_n z^n / n! = \exp \sum_1 R_n z^n / n!,$$

or

$$P_n = Y_n(R_1, R_2, \ldots, R_n)$$

$$= Y_n(P^c(1), P^c(2), \ldots, P^c(n)),$$

as in Theorem 8.

For convenience we provide the following table. These values were obtained by Wright [18] with the help of a computer. The last two rows of the table below were independently obtained by Evans, Harary, and Lynn [9] in terms of transitive digraphs.

TABLE

| n | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $\bar{P}^c(n)$ | 1 | 1 | 3 | 10 | 44 | 238 | 1650 |
| $\bar{Q}^c(n)$ | 1 | 2 | 6 | 21 | 94 | 521 | 3485 |
| $\bar{P}(n)$ | 1 | 2 | 5 | 16 | 63 | 318 | 2045 |
| $\bar{Q}(n)$ | 1 | 3 | 9 | 33 | 139 | 718 | 4535 |
| $P^c(n)$ | 1 | 2 | 12 | 146 | 3060 | 101642 | 5106612 |
| $Q^c(n)$ | 1 | 3 | 19 | 233 | 4851 | 158175 | 7724333 |
| $P(n)$ | 1 | 3 | 19 | 219 | 4231 | 130023 | 6129859 |
| $Q(n)$ | 1 | 4 | 29 | 355 | 6942 | 209527 | 9535241 |

REMARK 5. It is easy to show that; (i) $P^c(n)$ is even for all $n > 1$, (ii) $Q^c(n)$ and $P(n)$ are odd for all $n > 1$ [14].



We give now an example which may help to familiarize the reader with the meaning of Theorem 8. For the sake of simplicity we let $n = 3$. Using the preceding Table and Table 3 [13, p. 49], we obtain

(i) $$\begin{aligned} P(3) &= Y_3(P^c(1), P^c(2), P^c(3)) \\ &= P^c(3) + 3P^c(2)P^c(1) + (P^c(1))^3 \\ &= 12 + 3\cdot 2\cdot 1 + (1)^3 \\ &= 19. \end{aligned}$$

(ii) $$\begin{aligned} P^c(3) &= Y_3(fP(1), fP(2), fP(3)) \\ &= f_1P(3) + f_2(3P(2)P(1)) + f_3(P(3))^3 \quad (f_k = (-1)^{k-1}(k-1)!) \\ &= 1\cdot 19 + (-1)(3\cdot 3\cdot 1) + 2(1)^3 \\ &= 19 - 9 + 2 \\ &= 12. \end{aligned}$$

(iii) $$\begin{aligned} Q(3) &= Y_3(Q^c(1), Q^c(2), Q^c(3)) \\ &= Q^c(3) + 3Q^c(2)Q^c(1) + (Q^c(1))^3 \\ &= 19 + 3\cdot 3\cdot 1 + 1 \\ &= 29. \end{aligned}$$

(iv) $$\begin{aligned} Q^c(3) &= Y_3(fQ(1), fQ(2), fQ(3)) \\ &= f_1Q(3) + f_2(3Q(2)Q(1)) + f_3(Q(1))^3 \quad (f_k = (-1)^{k-1}(k-1)!) \\ &= 1\cdot 29 + (-1)(3\cdot 4\cdot 1) + 2(1)^3 \\ &= 29 - 12 + 2 \\ &= 19. \end{aligned}$$

We make four remarks without proof. The following remarks give interrelation between finite topology and lattice theory (graph theory, semigroup theory).

REMARK 6. It follows from some basic results of lattice theory [2]; (i) $T(X)$ is a complete lattice, (ii) $T_0(X)$ is a meet-semilattice, and (iii) the number of nonisomorphic distributive lattice of length $n$ is equal to $\bar{T}_0(n)$.

REMARK 7. Evans, Harary, and Lynn [9] have shown that there is a 1-to-1 correspondence between elements of $T(X)$ and $D_G(X)$, the set of all trasitive diagraphs that can be defined on $X$.

REMARK 8. Let $B(n)$ be the semigroup of all binary relations on a set with $n$ elements, represented as $n \times n$ matrices over the Boolean algebra $B = \{0, 1\}$ of order 2. In [4] it was shown; (i) $T_0(n) = |E(n)|$ where $E(n)$ denotes the set of all non-singular idempotent matrices of $B(n)$, and (ii) $\bar{T}_0(n) = |D(n)|$ where $D(n)$ denotes the set of all non-singular D-classes of $B(n)$.

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