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THE LEVEL POLYNOMIALS OF THE FREE DISTRIBUTIVE LATTICES

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We show that there exist a set of polynomials $\{L_k \mid k = 0, 1 \cdots\}$ such that $L_k(n)$ is the number of elements of rank k in the free distributive lattice on n generators. $L_0(n) = L_1(n) = 1$ for all n and the degree of L_k is $k-1$ for $k \geq 1$. We show that the coefficients of the L_k can be calculated using another family of polynomials, P_j. We show how to calculate L_k for $k = 1, \ldots, 16$ and P_j for $j = 0, \ldots, 10$. These calculations are enough to determine the number of elements of each rank in the free distributive lattice on 5 generators a result first obtained by Church [2]. We also calculate the asymptotic behavior of the L_k's and P_j's.

1. Introduction

The question of enumerating FD(n), the free distributive lattice on n generators, for arbitrary n, was first posed by Dedekind in 1897 [4]. Many authors [2–13, 15–18] have concerned themselves with this problem. Exact answers are known only for $n \leq 6$: Dedekind ($n = 1$, 3; $n = 2$, 6; $n = 3$, 20; $n = 4$, 168) [4]; Church ($n = 5$, 7581) [2]; Ward ($n = 6$, 7,823,352) [16]. There is some question as to the value of |FD(7)| (see Church (2,414,682,040,998) [3] and Lunnon (2,208,061,288,138) [11]). Korshunov has just recently announced an asymptotic formula for |FD(n)| [10].

Following the analysis in Birkhoff [1], we consider FD(n) to be the set of all closed from below subsets of the power set of $\boldsymbol{n} = \{1, \ldots, n\}$. (For unfamiliar, undefined terms consult [1].) Thus an element of FD(n) has rank k if it consists of exactly k subsets of $\boldsymbol{n}$. Thus FD(n) has $2^n + 1$ levels. The polynomials L_k which we present in this paper enumerate the number of elements of rank k in FD(n). By symmetry, it is easy to see that for all k, n with $0 \leq k \leq 2^n$, $L_k(n) = L_{2^n - k}(n)$. Thus

$$|\text{FD}(n)| = 2\left(\sum_{j=0}^{2^{n-1}-1} L_j(n)\right) + L_{2^{n-1}}(n).$$

Our analysis will enable us to compute $L_k(5)$ for $k = 0; \ldots, 32$, and hence FD(5).

The L_k's are similar in nature to the chromatic polynomials associated with graphs. Furthermore, we show that the coefficients of the L_k's can be derived

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from another family of polynomials $\{P_j\}$. The nature of the P_j's is somewhat more involved than that of the L_k's. We described the asymptotic behavior of both the L_k's and the P_j's.

We introduce some notation which we will use throughout this paper. As we have already indicated, for an integer n, we use $\boldsymbol{n}$ to denote $\{1, 2, \ldots, n\}$. Note $\boldsymbol{0} = \emptyset$. For a set X we use $|X|$ to denote its cardinality and 2^X to denote its power set. For a real number r, we use $\lfloor r \rfloor$ to denote the greatest integer not exceeding r and $\lceil r \rceil$ the least integer not smaller than r.

The results in this paper were derived in the author's dissertation [12] and some were stated without proof in [13].

2. The polynomial nature of L_k

2.1. Theorem. *For $k \geq 1$*

$$L_k(n) = \sum_{t=a_k}^{k-1} C(t, k) \binom{n}{t}$$

where $a_k = \lfloor \log_2 k \rfloor$ and $C(t, k)$ is the number of closed from below subsets of cardinality k of $2^{\boldsymbol{t}}$ which contain all singleton subsets of $\boldsymbol{t}$. Clearly, $L_0(n) = L_1(n) = 1$ for all n.

Proof. For each $X \in \mathrm{FD}(n)$ with $|X| = k$, X contains some number t of singleton sets with $2^t \geq k$. A little reflection shows that once we choose t of the n singleton subsets of $\boldsymbol{n}$, we can embed the closed from below subsets of $2^{\boldsymbol{t}}$ of cardinality k which cover all the singleton subsets of $\boldsymbol{t}$ into $\mathrm{FD}(n)$. □

Remark. Some unpublished notes of the late R. Church which are in the author's possession prove that he was aware of Theorem 2.1 and give the values of L_k for $k = 0, \ldots, 12$ and some of the coefficients of L_k for $k = 13, \ldots, 16$. However, these notes do not exhibit any additional results about them and thus it is difficult to ascertain exactly what Church knew about the L_k.

2.2. Corollary. *For $k \geq 1$, $\deg L_k = k - 1$ and*

$$\lim_{n\to\infty} \frac{L_k(n)(k-1)!}{n^{k-1}} = 1.$$

Proof. This corollary follows from Theorem 2.1 and the observation that $C(k-1, k) = 1$ for all $k \geq 1$. $C(k-1, k) = 1$ because the only subset of $2^{\boldsymbol{k-1}}$ consisting of k sets and containing all the singleton sets is the set $\{\emptyset, \{1\}, \ldots, \{k-1\}\}$. □

Remark. Note that $0, 1, \ldots, a_k - 1$ are among the roots of $L_k(n)$. These are the only possible non-negative integral roots of $L_k(n)$, since if $n \geq a_k$, there exists at

least one closed from below subset of $2^{\boldsymbol{a_k}}$ having cardinality k. It is possible for $L_k(n)$ to have negative integers as roots, e.g. -1 is a root of $L_4(n)$ and -9 is a root of $L_5(n)$. All the $L_k(n)$ up to $k = 7$ have only real roots, each with multiplicity one. Whether this is true in general is not known to the author.

Our next goal is to show that the coefficients (the $C(t, k)$) of the L_k's can be calculated from a family of polynomials. We first introduce some notation.

Notation. Let $\mathscr{C}(t, k)$ denote the collection of all closed from below subsets of $2^{\boldsymbol{t}}$ which have cardinality k and contain all singleton subsets of $\boldsymbol{t}$. Thus $C(t, k) = |\mathscr{C}(t, k)|$. We let $\mathscr{P}_j(k) = \mathscr{C}(k-j-1, k)$ and $P_j(k) = C(k-j-1, k)$. Furthermore, we let $\mathscr{C}_1(a, b) = \{S \in C(x, b)|$ such that no singleton set is a maximal element of $S\}$ and $C_1(a, b) = |\mathscr{C}_1(a, b)|$. Note that we use a_k to denote $\lceil \log_2 k \rceil$.

Remark. Note that

$$L_k(n) = \sum_{j=0}^{k-a_k-1} P_j(k) \binom{n}{k-j-1}.$$

Thus P_0 gives the first coefficient of L_k, etc. Note also that $P_0(k) = C(k-1, k) = 1$ for all $k \geq 0$. We will soon show that for a fixed j, $P_j(k)$ is a polynomial in k of degree $2j$.

3. The polynomial nature of P_j

Notation. Let Δ be a collection of sets and m an integer. We use $S(m, \Delta)$ to denote $|\{T \in \Delta \mid |T| = m\}|$.

3.1. Lemma. *$C_1(a, b) \neq 0$ if and only if $2^a \geq b \geq a + 1 + \lfloor \frac{1}{2}(a+1) \rfloor$. Furthermore, $\Delta \subset \mathscr{C}(a, a + 1 + \lfloor \frac{1}{2}(a+1) \rfloor)$ implies that $S(m, \Delta) = 0$ for $m \geq 3$ and consequently $S(2, \Delta) = \lfloor \frac{1}{2}(a+1) \rfloor$.*

Proof. Clearly $b > 2^a$ implies that $C_1(a, b) = 0$, since 2^a cannot contain any closed from below subset which has more than 2^a elements.

Let m_a be the smallest integer such that $C_1(a, m_a) \neq 0$. Suppose $m_a < 2^a$. If $X \in \mathscr{C}_1(a, m_a)$, then by adding a minimal element of $2^a - X$ to X we see that $C_1(a, m_a + 1) \neq 0$. Continuing in this way we see that $C_1(a, b) \neq 0$ for all $m_a \leq b \leq 2^a$.

Thus we need only show that $m_a = a + 1 + \lfloor \frac{1}{2}(a+1) \rfloor$. Clearly we can cover all the a singletons of 2^a by $\lfloor \frac{1}{2}(a+1) \rfloor$ two element subset of $\boldsymbol{a}$. Thus $m_a \leq a + 1 + \lfloor \frac{1}{2}(a+1) \rfloor$. It is clear that $\lfloor \frac{1}{2}(a+1) \rfloor$ is the smallest number of two element subsets of $\boldsymbol{a}$ which contain all the singletons of 2^a. If $\Delta \in \mathscr{C}(a, b)$ for some b, it follows that for each singleton $\{x\}$ of 2^a there exists some two element subset

$u_x \in \Delta$ such that $\{u\} \leqslant u_x$. Thus $S(2, \Delta) \geqslant \lfloor \frac{1}{2}(a+1) \rfloor$ and hence $|\Delta| \geqslant a+1+\lfloor \frac{1}{2}(a+1) \rfloor$, which proves all the various remaining claims made by the lemma. □

3.2. Theorem. *For* $j \geqslant 0$,

$$P_j(k) = \sum_{i=m_j}^{2j} C_1(i, i+j+1) \binom{k-j-1}{i}.$$

where m_j *is the smallest integer such that*

$$2^{m_j} \geqslant m_j + j + 1 > 2^{m_j - 1}.$$

Proof. Let $\Delta \in \mathcal{P}_j(k)$. This means that $\sum_{i=2}^{k-j-1} S(i, \Delta) = j$. Let α_Δ be the number of singletons in Δ which are not maximal elements.

From the proof of Lemma 3.1 it follows that $\alpha_\Delta \leqslant 2j$. Since each nonempty non-singleton member of Δ must be some union of the non-maximal singletons of Δ, it follows that $j \leqslant 2^{\alpha_\Delta} - \alpha_\Delta - 1$, and hence that $\alpha_\Delta \geqslant m_j$. Thus every element of $\mathcal{P}_j(k)$ can be constructed as follows. We choose i elements $a_1, \ldots, a_i$ from $\mathbf{k-j-1}$ (where $m_j \leqslant i \leqslant 2j$) and pick some element Z from $\mathscr{C}_1(i, i+j+1)$. Pick the obvious bijection f between $\mathbf{i}$ and $\{a_1, \ldots, a_i\}$ and let $\bar{f}: 2^i \to 2^{\{a_1, \ldots, a_i\}}$ be the bijection induced by f. Let $\Delta = \bar{f}(Z) \cup \mathbf{k-j-1}$. Then $\Delta \in \mathcal{P}_j(k)$, and every element of $\mathcal{P}_j(k)$ can be constructed in this way. Hence the theorem follows. □

3.3. Corollary. *For* $j \geqslant 0$, $P_j(k)$ *is of degree* $2j$ *and has leading coefficient equal to* $1/(2^j j!)$. *Thus asymptotically,* $P_j(k)$ *is* $k^{2j}/(2^j j!)$.

Proof. This result follows from Theorem 3.2 if we show that $C_1(2j, 3j+1) = (2j)!/(2^j j!)$. To find out how many distinct collections of j 2-element subsets of a $2j$ element set, T, cover all singleton subsets we proceed as follows. Consider all $(2j)!$ permutations of T and read off two consecutive elements at a time from left to right and take them as the j 2-element subsets. Clearly, every collection can be obtained in this way with multiplicity $2^j j!$ since the order in which we pick each of the elements in a 2-element set and the order in which we pick the 2-element subsets are irrelevant. □

Remarks. The next section will present a reasonably straightforward approach to calculating P_j for $j = 0, \ldots, 10$ and will present some additional information on the leading coefficients of P_j.

4. Calculating P_j

4.1. Lemma. *Let* $X \in \mathcal{P}_j(k)$. $S(q, X) > 0$ *implies that* $2^q - q - 1 \leqslant j$.

Proof. If $S(q, X) > 0$, then certainly there exist at least $2^q - q - 1$ nonempty non-singleton elements in X. But j is the exact number of nonempty non-singleton elements in X. □

4.2. Theorem. *For* $k \geqslant 0$ *we have that:*

(1) $P_0(k) = 1$;

(2) $$P_1(k) = \binom{\binom{k-2}{2}}{1} = \binom{k-2}{2};$$

(3) $$P_2(k) = \binom{\binom{k-3}{2}}{2} = 3\binom{k-3}{4} + 3\binom{k-3}{3};$$

(4) $$P_3(k) = \binom{\binom{k-4}{2}}{3} = 15\binom{k-4}{6} + 30\binom{k-4}{5} + 16\binom{k-4}{4} + \binom{k-4}{3}$$

(5) $$P_4(k) = \binom{\binom{k-5}{2}}{4} + \binom{k-5}{3} = 105\binom{k-5}{8} + 315\binom{k-5}{7} + 330\binom{k-5}{6}$$
$$+ 135\binom{k-5}{5} + 15\binom{k-5}{4} + \binom{k-5}{3};$$

(6) $$P_5(k) = \binom{\binom{k-6}{2}}{5} + \binom{k-6}{3}\left(\binom{k-6}{2} - 3\right);$$

(7) $$P_6(k) = \binom{\binom{k-7}{2}}{6} + \binom{k-7}{3}\binom{\binom{k-7}{2} - 3}{2};$$

(8) $$P_7(k) = \binom{\binom{k-8}{2}}{7} + \binom{k-8}{3}\binom{\binom{k-8}{2} - 3}{3} + 6\binom{k-8}{4};$$

(9) $$P_8(k) = \binom{\binom{k-9}{2}}{8} + \binom{k-9}{3}\binom{\binom{k-9}{2} - 3}{4} + 6\binom{k-9}{4}\binom{\binom{k-9}{2} - 5}{1}$$
$$+ 15\binom{k-9}{5} + 20\binom{k-9}{6};$$

$$(10)\quad P_9(k)=\binom{\binom{k-10}{2}}{9}+\binom{k-10}{3}\binom{\binom{k-10}{2}-3}{5}+6\binom{k-10}{4}\binom{\binom{k-10}{2}-5}{2}$$
$$+15\binom{k-10}{5}\binom{\binom{k-10}{2}-6}{1}$$
$$+20\binom{k-10}{6}\binom{\binom{k-10}{2}-6}{1}+4\binom{k-10}{4};$$

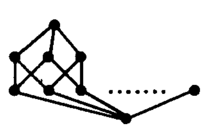
$$(11)\quad P_{10}(k)=\binom{\binom{k-11}{2}}{10}+\binom{k-11}{3}\binom{\binom{k-11}{2}-3}{6}+6\binom{k-11}{4}\binom{\binom{k-11}{2}-5}{3}$$
$$+15\binom{k-11}{5}\binom{\binom{k-11}{2}-6}{2}+20\binom{k-11}{6}\binom{\binom{k-11}{2}-6}{2}$$
$$+4\binom{k-11}{4}\binom{\binom{k-11}{2}-6}{1}$$
$$+60\binom{k-11}{5}+10\binom{k-11}{5}+\binom{k-11}{4}.$$

Note that for $0\leqslant j\leqslant 5$ we have expanded $P_j(k)$ in the form of Theorem 3.2.

Proof. What we basically do is investigate all the "closed from below"-like structures which it is possible to erect on $k-j-1$ singletons which consist of j elements. (1) is trivial. We note that by Lemma 4.1, if $1\leqslant j\leqslant 3$, $S(3,X)=0$ for all $X\in\mathcal{P}_j(k)$. Hence in these cases $S(2,X)=j$, and the result is immediate. Also from Lemma 4.1 it follows that if $4\leqslant j\leqslant 10$, $S(4,X)=0$ for all $X\in\mathcal{P}_j(k)$.

We observe that any structure which can be erected with j elements can be erected whenever we are allowed to construct structures with $j+1$ elements, in the sense that we can preserve the identical elements and simply add on another two element set. Thus, as we go from j to $j+1$ we need only discover the fundamentally new structures which become possible. This will become clearer shortly as we apply this principle. The reader might get an idea of what is being discussed by closely examining the forms of the $P_j(k)$'s as they appear in the statement of this theorem.

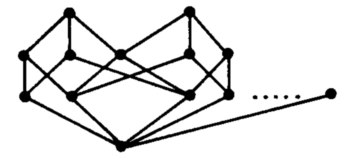
Thus in going from $j=3$ to $j=4$, we can still erect a structure having only two elements sets. This gives us the first term. However, we have a "new structure" which is possible, namely,



$k-5$ singletons

We note that in order for $S(3,X)\geqslant 2$, we must have $j\geqslant 7$. Hence no "new structures" appear for $j=5,6$, and we get $P_5(k)$ and $P_6(k)$ simply by adding on additional two element sets.

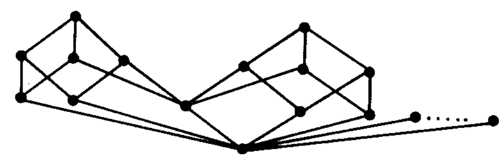
For $j=7$, a "new structure" appears:



$k-8$ singletons

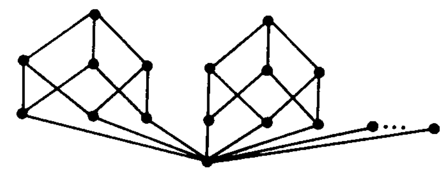
This contributes the term $6\binom{k-8}{4}$, since for every choice of 4 singletons, there are 6 ways to erect this structure. All other terms are holdovers, with additional two element sets.

For $j=8$,



$k-9$ singletons

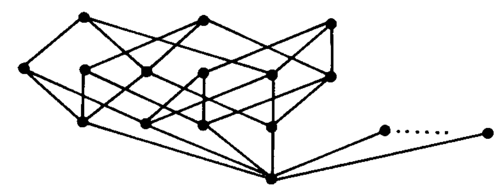
and



$k-9$ singletons

make their appearances, contributing $15\binom{k-9}{5}$ and $20\binom{k-9}{6}$ respectively.

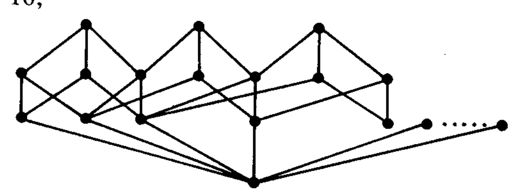
For $j=9$,

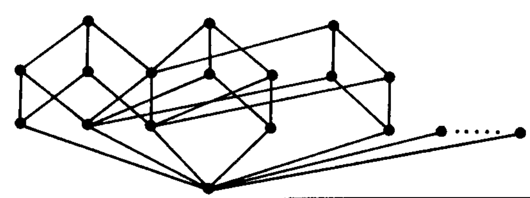


$k-10$ singletons

appears, contributing $4\binom{k-10}{4}$.

Finally for $j=10$,





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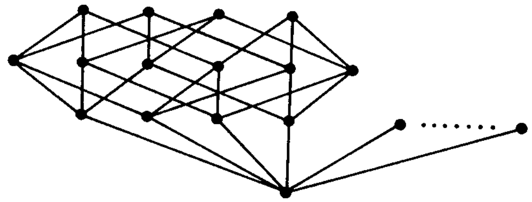

(all with $k-11$ singletons) appear, contributing $60\binom{k-11}{5}$, $10\binom{k-11}{5}$, and $\binom{k-11}{4}$ respectively. □

Remark. We were not just fortunate, in that we were entirely able to dispense working with any sets X such that $S(4, X)>0$. In general, if we wish to calculate FD(n), we need to know $L_k(n)$ for $0\leq k\leq 2^{n-1}$. Since $C(n-1, 2^{n-1})=1$, we need only to be able to calculate $P_{2^{n-1}-n-1}(k)$. But by Lemma 4.1, we see that $S(n-1, X)=0$ for all suitable X since

$$2^{n-1}-(n-1)-1=2^{n-1}-n\not\leq 2^{n-1}-n-1=j.$$

Thus to calculate FD(n), we essentially only need to consider closed from below subsets having no elements of cardinality greater than $n-2$.

It is interesting to note that Lunnon [11] calculates the values for FD(n), via a two stage reduction, so that he need only consider various properties of elements of FD($n-2$) in order to calculate FD(n). He asserts [11; p. 178, line 4] that a three stage reduction produces nothing useful.

The reader will also note that we have achieved a two stage reduction, i.e., we need only work with sets of cardinality $\leq n-2$ to calculate FD(n), via the $L_k(n)$'s $C(t, k)$'s, $P_j(k)$'s and $C_1(a, b)$'s. However, in trying to push the reduction one step further we ran into trouble. For one thing, it is no longer clear what $C_2(c, d)$, i.e., the proper extension of the sequence $C(t, k)$ and $C_1(a, b)$ should be.

Note that the argument of Theorem 4.2 can be generalized to yield another proof of Corollary 3.3. We conclude this section by giving closed formulas for the first ten coefficients of P_j for all $j\geq 0$. The derivations of these results can be found in [12]. We give some preliminary material first.

Notation. Let p, q, r be natural numbers. Let $T\subset \mathbf{p}$, and $S\subset 2^{\mathbf{p}}$ be such that $X\in S$ implies that $|X|=r$. By $Z(p, S, q, r, T)$ we mean

$$|\{\{\alpha_1,\ldots,\alpha_q\}|\ \alpha_i\subset \mathbf{p}, |\alpha_i|=t \text{ for all } i\in \mathbf{q}, \bigcup_1^q \alpha_i\supset T,$$
$$\{\alpha_1,\ldots,\alpha_q\}\cap S=\emptyset, \text{ and } \alpha_i\neq\alpha_j \text{ if } i\neq j\}|.$$

If $S\subset 2^{\mathbf{p}}$ and $\gamma\subset \mathbf{p}$, by S_γ we mean $S\cap 2^\gamma$. Note $S_{\mathbf{p}}=S$.

The following result is just a straightforward application of the principle of inclusion and exclusion (see [14]).

4.3. Proposition. *Let p, q, r, S and T be as above, then*

$$Z(p, S, q, r, T)=\sum_{j=0}^{|T|}(-1)^j\sum_{\substack{\alpha\subset T\\ |\alpha|=j}}\binom{\binom{p-j}{r}-|S_{\mathbf{p}-\alpha}|}{q}$$

4.4. Theorem. (1) *For all j and $m=0, 1,\ldots, 4$, we have*

$$C_1(2j-m,\ 3j-m+1)=Z(2j-m, \emptyset, j, 2, \mathbf{2j-m}).$$

(2) *For all j and $m=5,\ldots, 9$, we have*

$$C_1(2j-m, 2j-m+1)=Z(2j-m, \emptyset, j, 2, \mathbf{2j-m})$$
$$+\binom{2j-m}{3}Z(2j-m, \Delta^*, j-4, 2, \mathbf{2j-m}-\Delta)$$

where $\Delta\subset\mathbf{2j-m}$ is some three element set and Δ^* is the set of all two element subsets of Δ.

Remarks. Note that the quantity $Z(p, S, q, r, T)$ is not simple to evaluate in general. The argument of Corollary 3.3 shows that $Z(2j, \emptyset, j, 2, \mathbf{2j})=(2j)!/2^j j!$. A similar argument shows that $Z(2j-1, \emptyset, j, 2, \mathbf{2j-1})=(2j)!(j-1)/2^j j!$ which enables us to write down the second coefficient of any P_j.

5. The L_k and FD(5)

We conclude by deriving $L_k(n)$ for $n=0,\ldots, 16$ and showing how to calculate the number of elements in FD(5) by rank (this result is first due to R. Church [2]).

5.1. Theorem. *For $n\geq 0$,*

(1) $L_0(n)=1;$

(2) $L_1(n)=1;$

(3) $L_2(n)=\binom{n}{1}=n;$

(4) $L_3(n)=\binom{n}{2}=(n^2-n)/2;$

(5) $L_4(n)=\binom{n}{2}+\binom{n}{3}=(n^3-n)/6;$

(6) $L_5(n)=3\binom{n}{3}+\binom{n}{4}=(n^4+6n^3-25n^2+18n)/24;$

(7) $L_6(n)=3\binom{n}{3}+6\binom{n}{4}+\binom{n}{5}=(n^5+20n^4-85n^3+100n^2-36n)/120;$

$$L_7(n)=\binom{n}{3}+15\binom{n}{4}+10\binom{n}{5}+\binom{n}{6}; \tag{8}$$

$$L_8(n)=\binom{n}{3}+20\binom{n}{4}+45\binom{n}{5}+15\binom{n}{6}+\binom{n}{7}; \tag{9}$$

$$L_9(n)=19\binom{n}{4}+120\binom{n}{5}+105\binom{n}{6}+21\binom{n}{7}+\binom{n}{8}; \tag{10}$$

$$L_{10}(n)=18\binom{n}{4}+220\binom{n}{5}+445\binom{n}{6}+210\binom{n}{7}+28\binom{n}{8}+\binom{n}{9}; \tag{11}$$

$$L_{11}(n)=13\binom{n}{4}+332\binom{n}{5}+1{,}385\binom{n}{6}+1{,}330\binom{n}{7}+378\binom{n}{8}+36\binom{n}{9}+\binom{n}{10}; \tag{12}$$

$$L_{12}(n)=10\binom{n}{4}+420\binom{n}{5}+3{,}243\binom{n}{6}+6{,}020\binom{n}{7}+3{,}276\binom{n}{8}+630\binom{n}{9}+45\binom{n}{10}+\binom{n}{11}; \tag{13}$$

$$L_{13}(n)=6\binom{n}{4}+500\binom{n}{5}+6{,}325\binom{n}{6}+20{,}979\binom{n}{7}+20{,}531\binom{n}{8}+7{,}140\binom{n}{9}+990\binom{n}{10}+55\binom{n}{11}+\binom{n}{12}; \tag{14}$$

$$L_{14}(n)=4\binom{n}{4}+560\binom{n}{5}+10{,}925\binom{n}{6}+59{,}619\binom{n}{7}+99{,}680\binom{n}{8}+58{,}989\binom{n}{9}+14{,}190\binom{n}{10}+1{,}485\binom{n}{11}+66\binom{n}{12}+\binom{n}{13}, \tag{15}$$

$$L_{15}(n)=\binom{n}{4}+600\binom{n}{5}+17{,}345\binom{n}{6}+145{,}050\binom{n}{7}+393{,}540\binom{n}{8}+379{,}764\binom{n}{9}+149{,}115\binom{n}{10}+26{,}235\binom{n}{11}+2{,}145\binom{n}{12}+78\binom{n}{13}+\binom{n}{14}; \tag{16}$$

$$L_{16}(n)=\binom{n}{4}+616\binom{n}{5}+25{,}945\binom{n}{6}+314{,}405\binom{n}{7}+1{,}313{,}260\binom{n}{8}+1{,}992{,}144\binom{n}{9}+1{,}226{,}799\binom{n}{10}+341{,}220\binom{n}{11}+45{,}760\binom{n}{12}+3{,}003\binom{n}{13}+91\binom{n}{14}+\binom{n}{15}. \tag{17}$$

Proof. For $k=0,\dots,15$ the values of $L_k(n)$ follow from Theorems 2.1 and 4.2. For $k=16$, the first 10 coefficients starting with $\binom{n}{15}$ can be calculated from Theorem 4.2. Note that the coefficient of $\binom{n}{4}$ is $C(4,16)$ which is clearly 1. □

Remark. Note that we can calculate the number of elements in FD(5) of a given rank from Theorem 5.1, since $L_{16+k}(5)=L_{16-k}(5)$ by the observation we made in the first section. Note also that we have the means at our disposal to calculate all the coefficients of $L_{17}(n)$. By Theorem 2.1,

$$L_{17}(n)=\sum_{t=5}^{16} C(t,17)\binom{n}{t}.$$

Theorem 4.2 allows us to calculate $C(t,17)$ for $t=6,\dots,16$. Note that $C(5,17)=L_{17}(5)=L_{15}(5)=605$, so that we know $L_{17}(n)$ entirely.

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