Book Review

The Golden Ratio

Reviewed by George Markowsky

The Golden Ratio
Mario Livio
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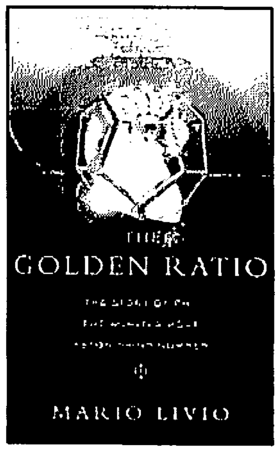

The number $(1+\sqrt{5})/2 = 1.618\ldots$ is widely known as the *golden ratio*, ϕ and phi. Phi appears in many different equations and formulas and has many interesting properties. Many people have heard marvelous tales about phi and how it permeates art and nature. My first exposure to phi was in a comic book entitled *Donald in Mathmagic Land*, which later became an animated cartoon seen by millions of people. As I grew up I kept seeing the same "facts" repeated in many different places, including popular books on mathematics, various mathematics textbooks, newspapers, and even in scholarly papers. It seemed as if everybody knew these basic "facts" about phi.

Around 1990 I decided to give a talk to the University of Maine Classics Club and thought that the golden ratio would be a fascinating topic for this audience. During the preparation of the talk I collected all of the usual stories about the golden ratio being used to design the Great Pyramid and the Parthenon, as well as about its aesthetic properties and its use by painters. I found the references to be quite vague, and in the process of trying to make my talk more precise, I actually began to look up measurements of buildings. Much to my surprise, the results did not support the claims that were being made about the golden ratio.

The results of my research were published in "Misconceptions about the Golden Ratio" (*The College Mathematics Journal*, Vol. 23, No. 1, Jan. 1992, 2–19). This paper debunks many of the more prominent claims about phi and documents their pervasive presence in the mathematical literature. For example, the name "golden ratio" is a nineteenth-century creation and is not an ancient name for phi. Furthermore, it does not appear that phi was used to design either the Great Pyramid or the Parthenon. For example, the Parthenon is 228 feet and 1/8 inch long, 101 feet and 3.75 inches wide, and 45 feet and 1 inch high. Taking the obvious ratios of length/width and width/height yields the number 2.25, which is quite far from phi, which is 1.618.... The number 2.25 = 9/4 is the ratio of two squares, and further study indicated that the gate to the Acropolis was built using this same ratio.

It also does not appear that Leonardo da Vinci used phi, nor is phi present in the proportions of the United Nations building in New York. Furthermore, in a large number of informal audience participation events, I have found that people do not pick golden rectangles more frequently than others (in fact, they are often picked less frequently than others), so that the statements about

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the aesthetic superiority of phi do not stand up to empirical tests. These claims and others are demolished in my paper in some detail.

Since publishing my paper, I have tried to get people, in particular mathematicians, to tell the truth about phi. Phi has many interesting mathematical properties that deserve to be brought to the public's attention. It is, however, a disservice to mathematics to mix the interesting properties of phi with dubious claims about its importance in art, architecture, human anatomy, and aesthetics.

Mario Livio's book, *The Golden Ratio*, is a broad survey of the properties of phi. The book is some 270 pages long, counting ten appendices, and bounces along describing various mathematical properties of phi, while at the same time trying to astonish the reader. It is the constant desire to astonish the reader that gets Livio into trouble and that is undoubtedly the source of the subtitle: *The Story of Phi, the World's Most Astonishing Number*. When I first heard about this book, I was hopeful that it would finally put many of the bogus stories about phi to rest, but unfortunately this book does not quite do so.

For example, in his discussion of the Parthenon, Livio quotes from my paper and gives proper attribution. However, I believe that he waffles on the issue of whether phi was used in the design of the Parthenon. On p. 74 he states:

> So, was the Golden Ratio used in the Parthenon's design? It is difficult to say for sure. While most of the mathematical theorems concerning the Golden Ratio (or "extreme and mean ratio") appear to have been formulated after the Parthenon had been constructed, considerable knowledge existed among the Pythagoreans prior to that.

I take strong issue with his conclusion that it is difficult to say for sure whether phi was used in the construction of the Parthenon. It seems to me that to even entertain the notion there has to be some reason to believe that it was true. It is clear that the Greeks were not as enamored of phi as people became once it received the name *golden ratio* in the nineteenth century. Calling phi division into mean and extreme ratio does not generate great excitement on the part of artists and architects. I have found no credible evidence that phi was ever used by Greek artists and architects for any project at all.

In Chapter 7 of his book Livio discusses the possible presence of phi in various paintings and its role in aesthetics. Again, he closely parallels my paper but does not cite the paper either in the text or in the notes to the text. In his discussion of Leonardo da Vinci, Livio reproduces exactly the painting and drawing discussed in my paper and analyzes them in the same manner. Later in the chapter he reproduces a diagram from my paper (he attributes the diagram to me, but does not give a reference) that shows forty-eight rectangles of different proportions that I have used a number of times to ask people which rectangle they find most pleasing.

On p. 183 of his book, Livio states:

> You can test yourself (or your friends) on the question of which rectangle you prefer best. Figure 84 shows a collection of forty-eight rectangles, all having the same height, but with their widths ranging from 0.4 to 2.5 times their height. University of Maine mathematician George Markowsky used this collection in his own informal experiments.

Interestingly, Livio does not reference my paper and does not quote my conclusion:

> In the experiments I have conducted so far, the most commonly selected rectangle is one with a ratio of 1.83.

Also, Livio does not point out that there are actually two golden rectangles in the diagram—one is oriented with the long dimension horizontal and the other with the long dimension vertical.

Livio could have performed a valuable service to the mathematical community had he written an accurate book about phi that treated it in a balanced manner and that consistently and thoroughly debunked the various misconceptions about phi that continue to circulate. Throughout this book, Livio struggles with the problem of wanting to "amaze" the public without going too far and losing respectability, but unfortunately he does not succeed in solving it.

He deserves credit for surveying a wide range of sources about phi, but in my opinion he is very inconsistent in how he uses them. In some cases he does an effective job of debunking nonsense, but in others his debunking is halfhearted. In some cases he omits data that would be harmful to establishing phi as the "most astonishing" number. Unfortunately, he also seems interested in spawning some new myths.

For example, on p. 9 he discusses Salvador Dali's "Sacrament of the Last Supper". The first "fact" that we are presented with is that the canvas measures approximately 105.5 inches by 65.75 inches, which "are in a Golden Ratio to each other." The ratio 105.5/65.75 is approximately 1.605, which is close to, but not equal to phi. If it was important for the painting to have phi as the ratio of its width to height, why not use a canvas of size approximately 106 inches by 66 inches, which has a ratio of 1.606, which is even closer to phi? We are next told that:

> Perhaps more important, part of a huge dodecahedron (...) is seen floating above the table and engulfing it. ... As we shall see in Chapter 4, regular solids (like the cube) that can be precisely enclosed by a sphere with all their corners resting on the sphere, and the dodecahedron in particular, are intimately related to the Golden Ratio.

This paragraph is odd for a number of reasons. First, it seems to suggest that somehow the cube is related to phi. Fortunately, when one reads chapter 4, one learns that "The Golden Ratio, ϕ, plays a crucial role in the dimensions and symmetry properties of some Platonic solids." As one might expect, the cube is not one of these solids. Another oddity is that while phi is present in the various proportions of the dodecahedron, it is interesting to note that the dodecahedron in the painting is distorted by the perspective that Dali used. Thus the proportions that we see in the painting itself are not those of the dodecahedron. Livio makes no attempt to actually measure any of the dimensions or to relate what we see to phi. He then proceeds to ask: "Why did Dali choose to exhibit the Golden Ratio so prominently in this painting?" This is astonishing because he did not give any evidence that phi is present in any significant way or that Dali had any interest in displaying phi in his paintings. Since Dali wrote about his paintings, one would expect that he would have mentioned his use of phi if that was of importance to him.

Another way of expanding what it means to "use phi" is to take all applications of Fibonacci numbers as applications of phi. Of course, one can express the Fibonacci numbers in terms of powers of phi, but Livio, like most authors writing to astonish people, neglects to mention that representing the Fibonacci numbers in terms of phi would make it much harder to "use" them in many applications. In particular, one can use the Fibonacci numbers happily without ever knowing about phi. Most properties of the Fibonacci numbers are best derived from the recurrence relation $F_n = F_{n-1} + F_{n-2}$, rather than by using phi. The fact that Fibonacci numbers can be written in terms of phi is a special case of the much more general results available as part of the theory of linear recurrence equations with constant coefficients.

On p. 86 Livio notes that if one takes any two positive integers and forms a series in which each new term is the sum of the preceding two terms, then eventually the ratio of a term to the preceding term converges to phi. He holds this out as an amazing fact but does not mention that in general if one picks any linear recurrence to generate terms in such a sequence, one will find that consecutive terms converge to some ratio that depends only on the recurrence relation and not on the starting points. In particular, pick any two positive integers and use the formula $F_n = F_{n-1} + 2F_{n-2}$. One will find that eventually the ratio of consecutive terms will approach 2. In this regard, phi is no more amazing than just about any other number.

Livio devotes a fair amount of space to discussing Luca Pacioli and his work on the "Divine Proportion". Livio notes that Pacioli ends up recommending a system of proportions for art not based on phi, even after he spends a lot of time discussing phi. A favorite gambit of Livio is to ask rhetorical questions such as the one on p. 178: "Short of intellectual curiosity, for what reason would so many artists even consider employing the Golden Ratio in their works?" The placement of this question is interesting because it follows a long section generally showing that artists have not been using phi in their work in any significant way. Of course, part of the answer to the question is that people keep writing books and papers extolling the aesthetic virtues of phi. With so much being written about phi by "experts", many artists feel strong pressure to at least look at phi.

At times the book appeals to mysticism. It talks about the "mystical" properties of integers and repeats a lot of nonsense about 666, the number of the beast; there is even a ridiculous formula relating 666 and phi. In particular, we are expected to be amazed (p. 23) that $\sin 666^\circ + \cos(6 \times 6 \times 6)^\circ$ is a "good approximation" of the negative of phi. Doing some "research" of this type, I was amazed to find that $\tan 666^\circ + \tan 666^\circ$ (about -2.75276) is "sort of" a good approximation of $-e$. In his discussion of pyramidology, Martin Gardner shows how in the absence of any rules one can torture numbers to come up with just about any result one wants.

Livio describes the rectangle construction that phi enthusiasts are so fond of (pp. 85-86). The fact that one gets a spiral of rectangles is considered amazing. Of course, one can do the same thing with any rectangle by dividing it into two pieces: a smaller rectangle similar to the original rectangle, and another rectangle that always has some fixed proportion. One can then create a spiral of smaller rectangles that converges to a point. Why the spiral derived from phi should be called "the Eye of God" is not explained. It is also not mentioned that the rectangle having dimensions 2 by $\sqrt{2}$ is even more amazing, since if one divides the long side in half one gets two rectangles similar to the original rectangle instead of just one rectangle and a square, as one does with phi.

In addition to its tendency to exaggerate the "uses" of phi, the book contains outright errors. For example, on p. 19 we are told that "we could even argue theoretically that the fact that 13 is a prime number, divisible only by 1 and itself, gives it an advantage over 10, because most fractions would be irreducible in such a system." Given that divisibility properties are independent of the base, this statement makes no sense.

On p. 116 the book says that "Jacques Bernoulli's association with the Golden Ratio comes through another famous curve." The curve referred to is the logarithmic spiral. However, the definition of the logarithmic spiral does not depend on phi, as can be seen from its equation $r = ae^{c\theta}$ in polar coordinates, where $a > 0$ and $c > 0$. This curve spirals infinitely often in both directions if $-\infty < \theta < \infty$. If one permits $c = 0$ then one gets a circle. One can certainly use phi as a parameter, but clearly one can also use any other positive number as a parameter. It is seriously misleading to claim that the properties of the logarithmic spiral somehow depend on phi. Even though the book has ten mathematical appendices that contain formulas, nowhere in the book does the formula for the logarithmic spiral appear. Of course, the formula for the logarithmic spiral would reveal that the curve has no special dependence on phi. The claims about the logarithmic spiral being related to phi are repeated at several places in the book.

Chapter 8 has some interesting material about tilings and quasi-crystals. It is a shame that this material is not developed with more technical details. Chapter 9, the final chapter, contains a long discussion about the unreasonable effectiveness of mathematics. The breezy way in which discussions of phi, the Fibonacci numbers, god, relativity, and string theory all roll into one another serves to glorify the role of phi.

The book also suffers from sloppy scholarship. In several places it follows my paper closely without giving any attribution. For example, a key point that I addressed in my paper (p. 5) was to develop some way of determining whether a measurement can actually be an indicator of the presence of phi. In my paper I proposed that people use a ±2% range around phi to at least treat the claim of the presence of phi as being worthy of consideration. I gave the rationale for this as follows:

> Another point overlooked by many golden ratio enthusiasts is the fact that measurements of real objects can only be approximations. Surfaces of real objects are not perfectly flat. Furthermore, it is necessary to specify the precision of any measurements and to realize that inaccuracies in measurements lead to greater inaccuracies in ratios. For example, a ±1% variation in the measurement of two lengths can lead to a roughly ±2% variation ($0.99/1.01 \approx 0.98$ to $1.01/0.99 \approx 1.02$) in the ratio that is computed. Thus someone eager to find the golden ratio somewhere can alter two numbers by ±1% and alter their ratio by roughly ±2%.

I was surprised to find the following discussion in Livio's book (p. 47) without attribution:

> The second point that is often ignored by the too-passionate Golden Ratio aficionados is that any measurements of lengths involve errors or inaccuracies. It is important to realize that any inaccuracy in length measurements leads to a yet larger inaccuracy in the calculated ratio. For example, imagine that two lengths, of 10 inches, each, are measured with a precision of 1 percent. This means that the result of the measurement of each length could be anywhere between 9.9 and 10.1 inches. The ratio of these measured lengths could be as bad as $9.9/10.1 = 0.98$, which represents a 2 percent inaccuracy—double that of the individual measurements. Therefore, an overzealous Golden Numberist could change two measurements by only 1 percent, thereby affecting the obtained ratio by 2 percent.

Even though Livio is aware of my paper and quotes it in various places, it is not even mentioned in the notes for the chapter where the preceding paragraph appears. This chapter also discusses the Great Pyramid and seems to follow the outlines of the discussion in my paper, again without any attribution. For example, compare p. 6 of my article with p. 56 of Livio's book. As in my paper he includes the link to Martin Gardner's discussion of pyramidology, which is the crank discipline of predicting the future by playing around with various measurements from the Great Pyramid. In my paper I pointed out that some of the "facts" that Martin Gardner used in his classic book, *Fads and Fallacies in the Name of Science*, to debunk the pyramidologists are actually based on their work. Again, no citations are given to my work. To his credit Livio concludes that the Golden Ratio was most likely not consciously incorporated in the design of the Great Pyramid.

I think that Livio lost a great opportunity. If he had focused on the mathematics of phi and spent less time on trying to astonish people with dubious claims, he would have done the mathematical community a great service. Given his ability to write, he would have also produced a much more interesting book.