# THE NUMBER OF MAXIMAL SUBGROUPS OF THE SEMIGROUP OF BINARY RELATIONS

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# THE NUMBER OF MAXIMAL SUBGROUPS OF THE SEMIGROUP OF BINARY RELATIONS

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It was shown in [11] that every finite group is the maximal subgroup of a semigroup $B_X$ of all binary relations on a finite set $X$. This result was extended by Plemmons and Schein [10] to the general case, using a theorem due to Zaretskii [14]. Clifford has given an entirely self-contained proof of this interesting result in [7].

The purpose of this paper is to count the number of maximal subgroups in a semigroup $B_X$ of all binary relations on a finite set $X$. This work has connection with previous work [2]-[5] by the first author, as well as from recent work of the second author [8]. In order to summarize our results, we find it expedient to first restate some of the definitions and notations.

Let $X=\{x_1, x_2, \cdots, x_n\}$ and assume $|X|=n>0$, where $|X|$ denotes the cardinality of a set $X$. A binary relation on $X$ is a subset of $X\times X$, and the set of all binary relations on $X$ is denoted by $B_X$. The product $\alpha\beta$ of two relations $\alpha$ and $\beta$ on $X$ is defined to be the relation

$$\alpha\beta=\{(a, b):(a, c)\in\alpha \text{ and } (c, b)\in\beta \text{ for some } c\in X\}$$

The operation is, of course, a generalization of the rule of composition of functions. Hence $B_X$ is a semigroup.

By a (0, 1)-matrix of order $n$ is meant an $n\times n$ matrix of 0's and 1's. Let $B_n$ denote the set of all such matrices. We consider the sum and product of members of $B_n$ to be the sum and product over the Boolean algebra $B=\{0, 1\}$ of order 2. Then $B_n$ is a semigroup under matrix multiplication, and the mapping

$$\alpha \longrightarrow A=(a_{ij})=\begin{cases}1 & \text{if } (x_i, x_j)\in\alpha\\ 0 & \text{otherwise}\end{cases}$$

is an isomorphism of $B_X$ onto $B_n$. Because of this correspondence, it is convenient to employ the same terminology and notation for the both the elements $\alpha\in B_X$ and the corresponding matrix $A\in B_n$. In the remainder of this paper we shall work with (0, 1)-matrices.

Let $V_n(B)$ denote the set of all $n$-tuples over $B$. The system $V_n(B)$ together with the operation of component-wise addition is called the (0, 1)-space of dimension $n$. A nonempty subset $W$ of $V_n(B)$ is said to be linearly independent if the zero $n$-tuple $z=(0, 0, \cdots, 0)$ is not in $W$ and no member of $W$ is a sum of other members of $W$. A subspace of $V_n(B)$ is a nonempty subset closed under addition. A linearly independent subset of $V_n(B)$ that generates a subspace $W$ of $V_n(B)$ under sum is called a basis of $W$. A basis always exists and is unique [2]. With $A \in B_n$ we associate the following sets: (i) the row space $R(A)=\{xA : x \in V_n(B)\}$, (ii) the column space $C(A)=\{Ax : x \in V^n(B)\}$ where $V^n(B)=\{y^t : y \in V_n(B)\}$ and $y^t$ denotes the transpose of $y$. For $A \in B_n$ the basis of the subspace of $V_n(B)$ generated by the nonzero rows of $A$ is called the row basis of $A$ and its cardinality is called the row rank of $A$. The definition of column basis and column rank are defined in a similar manner. Let $\rho_r(A)$ ($\rho_c(A)$) denote the row (column) rank of $A$. We remark that $A \in B_n$ need not have $\rho_r(A)=\rho_c(A)$ [2].

In order to discuss maximal subgroups of $B_n$, we shall need the following ideas. Two elements $a, b$ of an arbitrary semigroup $S$ are said to be $\mathscr{L}$($\mathscr{R}$)-equivalent iff they generate the same principal left (right) ideal in $S$. The relation $\mathscr{L} \cap \mathscr{R} = \mathscr{H}$, while the join $\mathscr{L} \vee \mathscr{R} = \mathscr{D}$. These equivalence relations are called Green's relations on $S$. The $\mathscr{L}$ ($\mathscr{R}$, $\mathscr{H}$, $\mathscr{D}$)-class containing $a$ will be denoted by $\mathscr{L}_a$($\mathscr{R}_a$, $\mathscr{H}_a$, $\mathscr{D}_a$). The Green's equivalences $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$, and $\mathscr{D}$ are discussed in [6] for an arbitrary semigroup $S$ and in [2], [8], [10]-[12], and [14] for the semigroup $B_n$. An element $e$ of an arbitrary semigroup $S$ is called idempotent if $e^2=e$. It is well known that if an $\mathscr{H}$-class contains an idempotent it is a group. Any two such $\mathscr{H}$-classes in the same $\mathscr{D}$-class are isomorphic and have the same cardinal number [6]. If some $a \in S$ is regular (i.e., $a \in aSa$), then each element of $\mathscr{D}_a$ is regular and $\mathscr{D}_a$ contains idempotents. In this case there is associated with $\mathscr{D}_a$ a subgroup of $S$, which is isomorphic to any $\mathscr{H}$-class in $\mathscr{D}_a$ that contains an idempotent [11]. Since each $\mathscr{H}$-class contains at most one idempotent [6], the enumeration of maximal subgroups of $B_n$ is equivalent to the enumeration of the idempotents of $B_n$. We define $D_k^n=\{$regular $\mathscr{D}$-classes of matrices $A \in B_n$ with $\rho_r(A)=k=\rho_c(A)\}$. We will determine $D_k^n$, where $0 \le k \le 4$, and then figure out the number of idempotents in each element of $D_k^n$.

In order to carry out this enumeration we need the following basic results. If $A \in B_n$, then $R(A)$ forms a lattice under set theoretic inclusion. The following basic results are due to Zaretskii [14].



THEOREM 1. *Let $A, B \in B_n$, then $A\mathscr{D}B$ iff $R(A)$ is lattice isomorphic to $R(B)$.*

THEOREM 2. *Let $A \in B_n$, then $A$ is regular iff $R(A)$ is a distributive lattice.*

It follows from these results and some basic results of lattice theory [1] that $|D_k^n|$ is equal to the number of nonisomorphic distributive lattices of length $k$. This number in turn is equal to the number of nonisomorphic partially ordered sets of $k$-elements [1]. Thus, $|D_k^n|$ depends only on $k$, and we denote $|D_k^n|$ by $m_k$. The fact that $|D_k^n|$ depends only on $k$ was also proven using a slightly different approach by the first author in [4]. By $D_{ki}^n$, $i=1, 2, \cdots, m_k$ we mean the individual elements of $D_k^n$. The following theorem will be useful in enumerating the subgroups of $B_n$.

THEOREM 3.

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $m_k$ | 1 | 1 | 2 | 5 | 16 | 63 | 318 | 2045 |

PROOF. The proof follows from the preceding remarks and the enumeration of partially ordered sets found in [1] and [13].

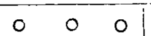
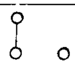
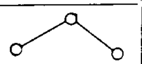
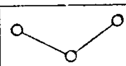
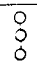
Let $G(n, k, i)$, $i=1, 2, \cdots, m_k$, denote the set of mutually isomorphic maximal subgroups of $D_{ki}^n$, and let $G(n, k)=\bigcup_{i=1}^{m_k} G(n, k, i)$. Thus $G(n, k)$ is the set of all maximal subgroups of $B_n$ contained in $D_k^n$. Finally let $G(n)$ denote the set of all maximal subgroups of $B_n$, then

$$|G(n)| = \sum_{k=0}^{n} \left(\sum_{i=1}^{m_k} |G(n, k, i)|\right)$$

From what has been said we get the answer to our problem, by examining the unique (up to isomorphism) lattice which is associated with each regular $\mathscr{D}$-class $D_{ki}^n$ obtained by considering the lattice which is obtained by considering the row space of any element of $D_{ki}^n$ as a lattice in a natural way.

Now to find $|G(n)|$, we focus on computing $|G(n, k, i)|$ for $0 \le k \le 4$ and $1 \le i \le m_k$. In the following tables, the column headed "Diagram" gives the Hasse diagram of the partially ordered set which is associated with the canonical lattice of $D_{ki}^n$, which depends only on $k$ and $i$. Let $o(G(n, k, i))$ denote the order of the groups in $G(n, k, i)$.

THEOREM 4. *For* $k=0$, 1, 2, *and* 3.

| $k$ | $i$ | *Diagram* | $\|G(n, k, i)\|$ | $o(G(n, k, i))$ |
|---|---|---|---|---|
| 0 | 1 | | 1 | 1 |
| 1 | 1 |  | $4^n-3^n$ | 1 |
| 2 | 1 |  | $(9^n-2(8^n)+7^n)/2$ | 2 |
| | 2 |  | $8^n-2(7)^n+6^n$ | 1 |
| 3 | 1 |  | $(18^n-3(17)^n+3(16)^n-15^n)/6$ | 6 |
| | 2 |  | $15^n-3(14)^n+3(13)^n-12^n$ | 1 |
| | 3 |  | $(15^n-3(14)^n+3(13)-12^n)/2$ | 2 |
| | 4 |  | $(15^n-3(14)^n+3(13)^n-12^n)/2$ | 2 |
| | 5 |  | $13^n-3(12)^n+3(11)^n-10^n$ | 1 |

PROOF. Since the number of maximal subgroups of $B_n$ is equivalent to the number of idempotents in $B_n$ we may identify $|G(n, k, i)|$ with $|E(D^n_{ki})|$. This set of results is proved in [3].

Before we continue we need some additional results. The following appears in [9].

THEOREM 5. *Let* $A\in B_n$, *then the Schutzenberger group associated with* $D_A$ *is isomorphic to the group of lattice isomorphisms of* $R(A)$.

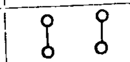
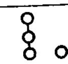
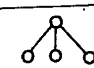
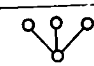
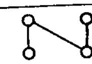
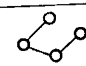
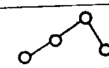
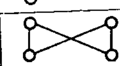
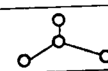
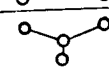
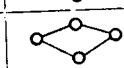
To help visualize this, we need the following result which appears in [8].

THEOREM 6. *There is a natural isomorphism between the group of automorphisms of a finite distributive lattice L and the group of automorphisms of the partially ordered set formed by the join-irreducible elements of L.*

The theorems above mean that for $A\in D^n_{ki}$, $|H_A|=o(G(n, k, i))$ depends only on $k$ and $i$, and in particular on the automorphisms of the partially ordered set of $k$ elements formed by the basis of $R(A)$.

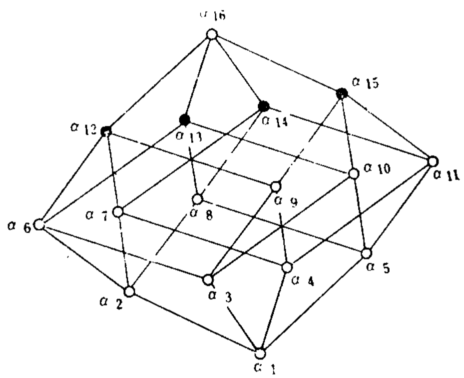


THEOREM 7. *For* $k=4$.

| $i$ | *Diagram* | $\|G(n, 4, i)\|$ | $o(G(n, 4, i))$ |
|---|---|---|---|
| 1 |  | $(35^n-4(34)^n+6(33)^n-4(32)^n+31^n)/24$ | 24 |
| 2 |  | $(28^n-4(27)^n+6(26)^n-4(25)^n+24^n)/2$ | 2 |
| 3 |  | $(26^n-4(25)^n+6(24)^n-4(23)^n+22^n)/2$ | 2 |
| 4 |  | $(26^n-4(25)^n+6(24)^n-4(23)^n+22^n)/2$ | 2 |
| 5 |  | $(23^n-4(22)^n+6(21)^n-4(20)^n+19^n)/2$ | 2 |
| 6 |  | $22^n-4(21)^n+6(20)^n-4(19)^n+18^n$ | 1 |
| 7 |  | $(28^n-4(27)^n+6(26)^n-4(25)^n+24^n)/6$ | 6 |
| 8 |  | $(28^n-4(27)^n+6(26)^n-4(25)^n+24^n)/6$ | 6 |
| 9 |  | $24^n-4(23)^n+6(22)^n-4(21)^n+20^n$ | 1 |
| 10 |  | $23^n-4(22)^n+6(21)^n-4(20)^n+19^n$ | 1 |
| 11 |  | $23^n-4(22)^n+6(21)^n-4(20)^n+19^n$ | 1 |
| 12 |  | $(26^n-4(25)^n+6(24)^n-4(23)^n+22^n)/4$ | 4 |
| 13 |  | $(22^n-4(21)^n+6(20)^n-4(19)^n+18^n)/2$ | 2 |
| 14 |  | $(22^n-4(21)^n+6(20)^n-4(19)^n+18^n)/2$ | 2 |
| 15 |  | $(22^n-4(21)^n+6(20)^n-4(19)^n+18^n)/2$ | 2 |
| 16 |  | $19^n-4(18)^n+6(17)^n-4(19)^n+15^n$ | 1 |

PROOF. Let us consider the partially ordered set represented by diagram (1) call it $\alpha$. By considering the ring of all subsets of $\alpha$ which are closed from below we get a distributive lattice [1]. In this case where the partially ordered set is the basis of the row space of a binary relation, the corresponding lattice, is the lattice which would be formed by the whole row space. Thus in this case we get the lattice $L_1$ represented by diagram (1′).

(1′)




As is proved in [8], the number of idempotents is equal to

$$\left(\sum_{i=0}^{k} (-1)^i \binom{k}{i} (\mathfrak{M}(A)-i)^n\right)/|H_A| \qquad (*)$$

where $A \in D^n_{ki}$ for some $i$ and $\mathfrak{M}(A)$ is the idempotent number of the matrix $A$. Thus we are left with the task of calculating the idempotent number $\mathfrak{M}(A)$, since all the other quantities in (*) may be readily calculated. We will carry out the calculations in details for $L_1$ and we have stated the result of the analogous calculations for the other lattices in the table above. In [8], a procedure is given for the calculation of $\mathfrak{M}(A)$, and we will briefly explain how it works, but will not present the proof here.

We first need to identify the meet-irreducible elements of $L_1$, and these are quite clearly the elements of $\alpha_{12}$, $\alpha_{13}$, $\alpha_{14}$, and $\alpha_{15}$. Let $w \in L_1$, $\theta = \{\alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{15}\}$. Following [8], we define $B(w) = \{x \in \theta : x \ngeq w\}$. Continuing we let $v_w$ be the join of all the elements in $B(w)$ (Note: if $B(w)$ is the empty set, then the join is the 0 element $\alpha_1$), and we let $q(w)$ be the number of elements in $L_1$ which are greater than or equal to $v_w$. We calculate the number $q(w)$ for each $w \in L_1$ and the number $\mathfrak{M}(A)$ is the sum of all the $q(w)$'s. Thus if $w = \alpha_{16}$ we get that $v_w = \alpha_{16}$ and consequently $q(w) = 1$. Similarly, if $w = \alpha_{15}$, we get $v_w = 16$, and $q(w) = 1$.



In fact, $q(w) = 1$ for all of the elements of $L_1$, except for $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_4$, and $\alpha_5$. For $w = \alpha_2$, $\alpha_3$, $\alpha_4$, $\alpha_5$, $q(w) = 2$, while $q(\alpha_1) = 16$. Thus in this case $\mathfrak{M}(A) = 11 + (4)(2) + 16 = 35$. Hence the number of maximal subgroups in $G(n, 4, 1)$ is equal to

$$(35^n - 4(34)^n + 6(33)^n - 4(32)^n + 31^n)/24,$$

since $o(G(n, 4, 1))$ is equal to 24.

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