The Number of Partially Ordered Sets. II.

By

K. Butler and G. Markowsky

The printer did not distinguish between $\mathcal{P}(n,k)$ - a set and $P(n,k) = |\mathcal{P}(n,k)|$

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THE NUMBER OF PARTIALLY ORDERED SETS. II.

By Kim Ki-Hang Butler* and George Markowsky

In this paper we extend some of the numerical results appearing in [2] dealing with certain classes of partially ordered sets. Techniques, more powerful than the matrix approach used in [2], are developed, which show that particular classes of partially ordered sets can be enumerated by polynomials of a certain type. Some of these polynomials are calculated, and then the enumeration of partially ordered sets is looked at from a graph theoretical point of view.

The reader should consult [2] for background and related results, as well as a more complete list of references. The reader might find [3] helpful, since some of the results in [2] are treated there along the lines similar to ones pursued in this paper. Basic properties and definitions relating to partially ordered sets can be found in [1].

Definition 1. Let $P(n, k)$ be the number of distinct (up to isomorphism) partial orderings on a set of n elements $\{x_1, x_2, \cdots, x_n\}$ such that there exist exactly k ordered pairs (x_i, x_j), $i \neq j$, for which $x_i > x_j$ in the given partial ordering. By $P(n, k)$ we mean a set consisting of one representative from each of the isomorphism classes counted by $P(n, k)$. Occasionally, we well abuse notion and think of $P(n, k)$ as consisting of equivalence classes. In any case, $P(n, k) = |P(n, k)|$.

Example. Thus $P(3, 2) = 2$, since the only possible posets are those corresponding to the following Hasse diagrams:

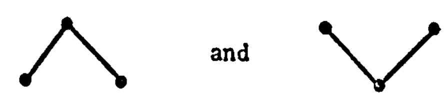

Diagram 1

Remark. Note that $P(n, k) = |D(n, k)|$ for all n and k, where $|D(n, k)|$ is the quantity defined in [2]. Also note that $P(n, k) = 0$ for $k > n(n-1)/2$.

Thus, if we let $P(n)$ denote the number of distinct (up to isomorphism) partial orderings which can be defined on n elements, then

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$$P(n)=\sum_{k=0}^{n(n-1)/2} P(n,k).$$

THEOREM 1. *Let* $k \geqq 0$ *be an integer.*

(*a*) *If* m, n *are integers and* $n \geqq m \geqq 0$, *then* $P(n,k) \geqq P(m,k)$.

(*b*) *If* m *is an integer and* $m \geqq 2k$, *then* $P(m,k)=P(2k,k)$.

Proof. (a) Pick an m element set $\{x_1, \cdots, x_m\}$. Extend it to an n element set $\{x_1, \cdots, x_m, x_{m+1}, \cdots, x_n\}$. Clearly, a partial ordering $\leqq$ on $x_1, \cdots, x_m$ can be extended to a partial ordering $\leqq^*$ on $x_1, \cdots, x_n$ by defining $x_i \leqq^* x_j$ iff (i) $i, j \leqq m$ and $x_i \leqq y_j$, or (ii) $i=j$. Clearly, the mapping

$$f: P(m, k) \Longrightarrow P(n, k)$$

given by $f(\leqq)= \leqq^*$ is an injection. Thus $P(n, k) \geqq P(m, k)$.

(b) From (1) we know that $P(m,k) \geqq P(2k,k)$. Let $\Delta \in P(m,k)$. Suppose Δ is a partial ordering on the set $\{x_1, \cdots, x_m\}$. Let

$S_\Delta=\{x_j$: there exists x_i such that $x_j <_\Delta x_i$ or $x_i <_\Delta x_j\}$.

Then $|S_\Delta| \leqq 2k$, since $\Delta \in P(m,k)$. Let $\Gamma=\{y_1, \cdots, y_{2k}\}$. Let

$$\Psi_\Delta: S_\Delta \Longrightarrow \Gamma$$

be an injection. Define $\Delta_{\psi_\Delta} \in P(2k,k)$ as follows:

$$y_i \leqq_{\Delta^{\psi_\Delta}} y_i \text{ for } i=1, \cdots, 2k$$

and

$$y_i <_{\Delta^{\psi_\Delta}} y_j \quad (i \neq j)$$

iff $y_j, y_j \in \Psi_\Delta(S_\Delta)$ and $\Psi_\Delta^{-1}(y_i) <_\Delta \Psi_\Delta^{-1}(y_j)$. The map

$$f: P(m,k) \Longrightarrow P(2k,k)$$

given by $f(\Delta)=\Delta_{\psi_\Delta}$, is injective. Hence, $P(m,k)=P(2k,k)$.

THEOREM 2. $P(n,4)=0$ *for* $n \leqq 3$, $P(4,4)=3$, $P(5,4)=10$, $P(6,4)=16$, $P(7,4)=18$, *and* $P(n,4)=19$ *for* $n \geqq 8$.

Proof. Clearly, $P(n,4)=0$ for $n \leqq 3$. From Theorem 1, we know that $P(n,4)=P(8,4)$ for $n \geqq 8$, and that for $4 \leqq n \leqq 8$ we have "natural" injections from $P(n,4)$ into $P(8,4)$. Thus we calculate $P(8,4)$ and in the process prove all the results presented above Let $\Delta \in P(8,4)$. This means that there exist exactly 4 ordered pairs (x_{i_k}, x_{j_k}), $k=1, \cdots, 4$ formed from $x_1, \cdots, x_8$ such that $x_{i_k} <_\Delta x_{j_k}$. Let $V_\Delta=\{(x_{i_1}, x_{j_1}), \cdots, (x_{i_4}, x_{j_4})\}$ and $T_\Delta=\{x_{j_1}, \cdots, x_{j_4}\}$. Let

$$\Pi: V_\Delta \Longrightarrow T_\Delta$$

be the projection on the second factor. Let $a_1, \cdots, a_{h_\Delta}, h_\Delta \leqq 4$, be the distinct elements of T_Δ. Then

$$|\Pi^{-1}(a_1)|, \cdots, |\Pi^{-1}(a_{h_\Delta})|$$

is a partition of 4. We will calculate $P(8,4)$, by considering the partitions of 4 and then calculating all Δ's which give rise to that particular partition. The partitions of 4 are: 4; 3, 1; 2, 2; 2, 1, 1; 1, 1, 1, 1. Corresponding to the partitions of 4 we have the partial orderings corresponding to the following Hasse diagrams.

Only (4), (7), and (4) can come from elements in $P(4,4)$ in the manner described

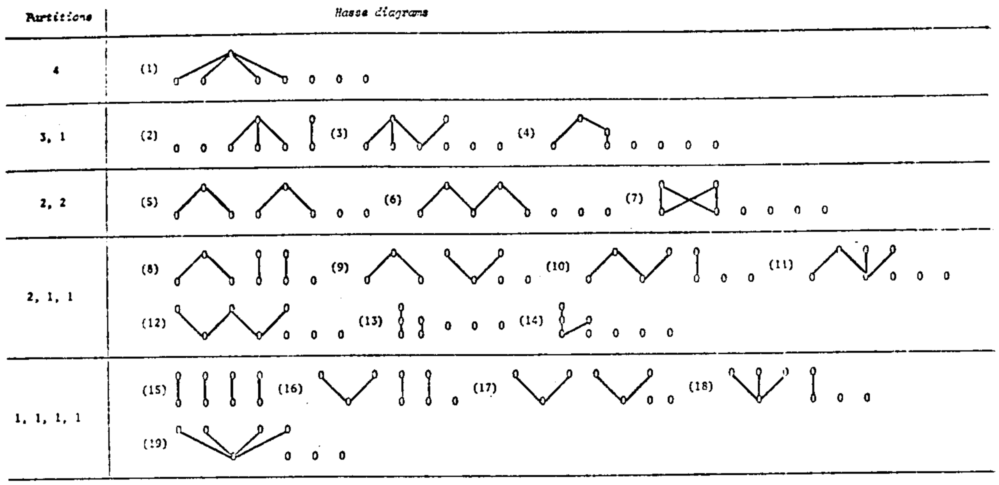


Diagram 2

in Theorem 1. Hence $P(4,4)=3$. The other results are derived similarly.

THEOREM 3. *$P(n,5)=0$ for $n \leqq 3$, $P(4,5)=3$, $P(5,5)=10$, $P(6,5)=25$, $P(7,5)=38$, $P(8,5)=44$, $P(9,5)=46$, and $P(n,5)=47$ for $n \geqq 10$.*

Proof. We shall not prove it, Since the proof is similar to the proof of Theorem 2. We will just list the Hasse diagrams of the posets corresponding to the various partitions of 5. See Diagram 3.

REMARK. For convenience we provide the following table which incorporates the

The numbers of nonisomorphic posets

n \ k	0	1	2	3	4	5
0	1	0	0	0	0	0
1	1	0	0	0	0	0
2	1	1	0	0	0	0
3	1	1	2	1	0	0
4	1	1	3	4	3	3
5	1	1	3	6	10	10
6	1	1	3	7	16	25
7	1	1	3	7	18	38
8	1	1	3	7	19	44
9	1	1	3	7	19	46
1	1	1	3	7	19	47
11	1	1	3	7	19	47

results of Theorems 2 and 3 as well as related results from [**2**, Theorem 9], which can be derived in the same way as the results above.

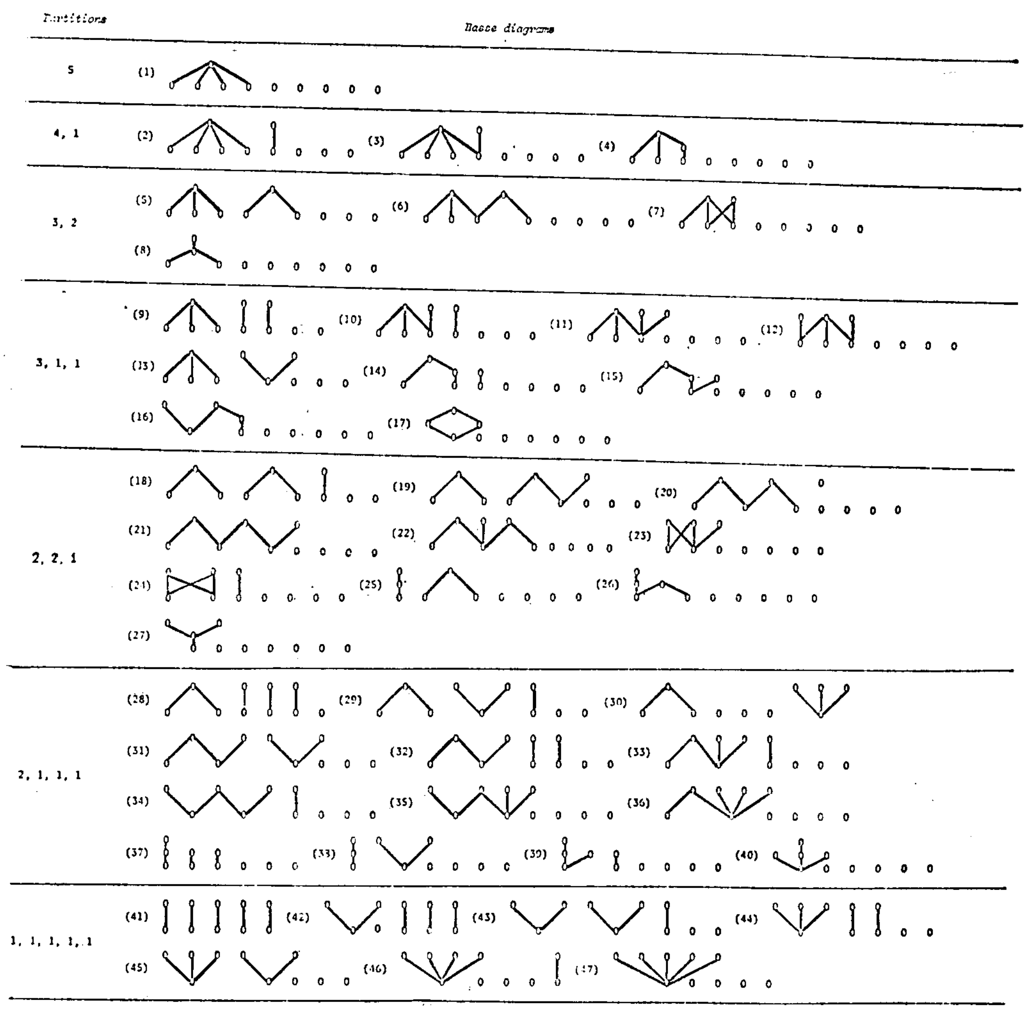


Diagram 3

Note that $P(4)=15+P(4,6)$. But clearly $P(4,6)=1$. Thus $P(4)=16$.

In [**2**, Theorem 9] it was shown that $P(n, \binom{n}{2})=1$, $P(n, \binom{n}{2}-1)=n-1$, and $P(n, \binom{n}{2}-2)=(n-1)(n-2)/2$.

The technique of Theorems 2 and 3 is not easily applicable in this case. We begin by stating and proving a general and interesting theorem.

THEOREM 4. *Let $k \geqq 0$ be an integer. Then, if n is such that $\binom{n}{2} \geqq k$, we have*

$$(+)\quad P\left(n, \binom{n}{2}-k\right)=\sum_{j=\lambda_k}^{2k} \sum_{\Delta \in P_1(j, \binom{j}{2}-k)} \binom{n-j+q_\Delta}{q_\Delta},$$

where λ_k is the smallest nonnegative integer such that $\binom{\lambda_k}{2} \geqq k$, q_Δ is as defined in the proof below, and

$$P_1\left(j, \binom{j}{2}-k\right)=\left\{\Delta \in P\left(j, \binom{j}{2}-k\right): x \in \Delta \Rightarrow \text{there exists } y \in \Delta \text{ such that } x \not\geqq y \text{ and } x \not\leqq y\right\}.$$

For fixed k the right hand of $(+)$ obviously contains only finitely many terms. Of course, if $k > \binom{n}{2}$, $P(n, \binom{n}{2}-k)=0$.

Proof. Let $\Delta \in P(n, \binom{n}{2}-k)$ and let $U_\Delta=\{\{x_{i_1}, x_{j_1}\}, \cdots, \{x_{i_k}, x_{j_k}\}\}$ be the k non-comparable pairs of elements in Δ. As usual we assume the underlying set of Δ is $\{x_1, \cdots, x_n\}$. Let $T_\Delta=\{x_{i_1}, \cdots, x_{i_k}, x_{j_1}, \cdots, x_{j_k}\}$. Clearly $|T_\Delta| \leqq 2k$. T_Δ is a poset in a natural way and $T_\Delta \in P_1(|T_\Delta|, \binom{|T_\Delta|}{2}-k)$. Consider the following undirected graph G_Δ: the vertices of G_Δ are the elements of T_Δ and the edges are the unordered pairs $\{x_{i_m}, x_{j_m}\}$, $m=1, \cdots, k$.

Let $C_{\Delta 1}, \cdots, C_{\Delta q_\Delta}$ be the connected components of G_Δ. Note that q_Δ is the number of connected components of G_Δ. For basic graph theoretical notions see [4]. (1) If $x_i \in C_{\Delta\beta}$ and $x_j \notin C_{\Delta\beta}$, then either $x_i > x_j$ or $x_i < x_j$. (2) If $x_i \in C_{\Delta\beta}$, $x_j \notin C_{\Delta\beta}$, and $x_j > x_i (x_j < x_i)$, then $x_j > x_h$ $(x_j < x_h)$ for all $x_h \in C_{\Delta\beta}$, since all elements in $C_{\Delta\beta}$ are incomparable in Δ. We know that $\Delta - T_\Delta$ is totally ordered and has $n-|T_\Delta|$ elements. From the above remarks, it follows that each of the components of G_Δ must be located either between two elements of $\Delta - T_\Delta$, or above all elements of $\Delta - T_\Delta$, or below all elements of $\Delta - T_\Delta$. More than one component can occupy the same position, but it is also clear from the above remarks that the components are naturally totally ordered by Δ, so that once the positions occupied by the components are chosen, they can be fitted in an unambiguous way. There are $n-|T_\Delta|+1$ possible positions of which we must choose q_Δ (repetitions allowed). Thus there are exactly

$$\binom{n-|T_\Delta|+q_\Delta}{q_\Delta}$$

elements (see [6]) in $P(n, \binom{n}{2}-k)$ which give rise to T_Δ via the process described at the beginning of the proof. But any $\Delta \in P(n, \binom{n}{2}-k)$ arises from some $T_\Delta \in P_1(j, \binom{j}{2}-k)$ for some $\lambda_k \leqq j \leqq 2k$, and hence the theorem follows.

REMARK. The technique used in Theorem 4 is very closely related to the technique used by the second author in obtaining some results dealing with the free distributive lattice in [5].

COROLLARY 1. *Let n and k be as in Theorem 4. Then $P(n, \binom{n}{2}-k)$ is a polynomial of degree k with leading coefficient $1/k!$. Thus for fixed k, $P(n, \binom{n}{2}-k)$ tends asymptotically $n^k/k!$ as $n \to \infty$. Of course this is only useful for calculating the tail end of the sequence $P(n, j)$ where $j=0, \cdots, \binom{n}{2}$.*

Proof. For each $\Delta \in P_1(j, \binom{j}{2}-k)$, $\lambda_k \leqq j \leqq 2k$, the term

$$\binom{n-|T_\Delta|+q_\Delta}{q_\Delta}$$

which appears in the right hand side of $(+)$ in Theorem 4 is of degree q_Δ in n. It is clear that each connected component of G_Δ contains at least 2 elements, and since G_Δ has at most $2k$ elements, $q_\Delta \leqq k$, for all Δ. $q_{\Delta_0}=k$ only in the case where Δ_0 is the unique element of $P_1(2k, \binom{2k}{2}-k)$, and the corollary follows immediately.

LEMMA 1.

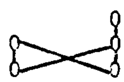
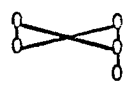
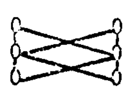
$P_1(j, \binom{j}{2} - k)$	Hasse diagrams
$P_1(3, 0)$	(Δ_1) o o o
$P_1(4, 3)$	(Δ_2)  (Δ_3) o 
$P_1(5, 7)$	(Δ_4)  (Δ_5) 
$P_1(6, 12)$	(Δ_6) 

Diagram 4

Proof. Deriving the results for $P_1(3,0)$, $P_1(4,3)$, and $P_1(6, 12)$ is straightforward. Suppose $\Delta \in P_1(5,7)$. A little reflection will show that U_Δ must have the form $\{\{a,b\}, \{a,c\}, \{d,e\}\}$. There are only two cases that need to be examined: $a > d$; $a < d$. If $a > d$,

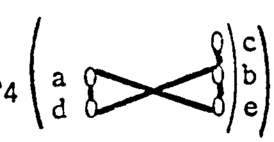
$\Delta \simeq \Delta_4$ 

Diagram 5

and if $a < d$, then $\Delta \simeq \Delta_5$.

THEOREM 5. $P(n, \binom{n}{2}-3) = (n^3-6n^2+23n-36)/6$, *for* $n \geqq 3$.

Proof. $\lambda_3=3$ and $q_{\Delta_1}=q_2=q_3=1$, $q_4=q_5=2$, $q_6=3$

Applying Theorem 4 we have

$P(n, \binom{n}{2}-3) = (n-2)+2(n-3)+2\binom{n-3}{2}+\binom{n-3}{3} = (n^3-6n^2+23n-36)/6.$

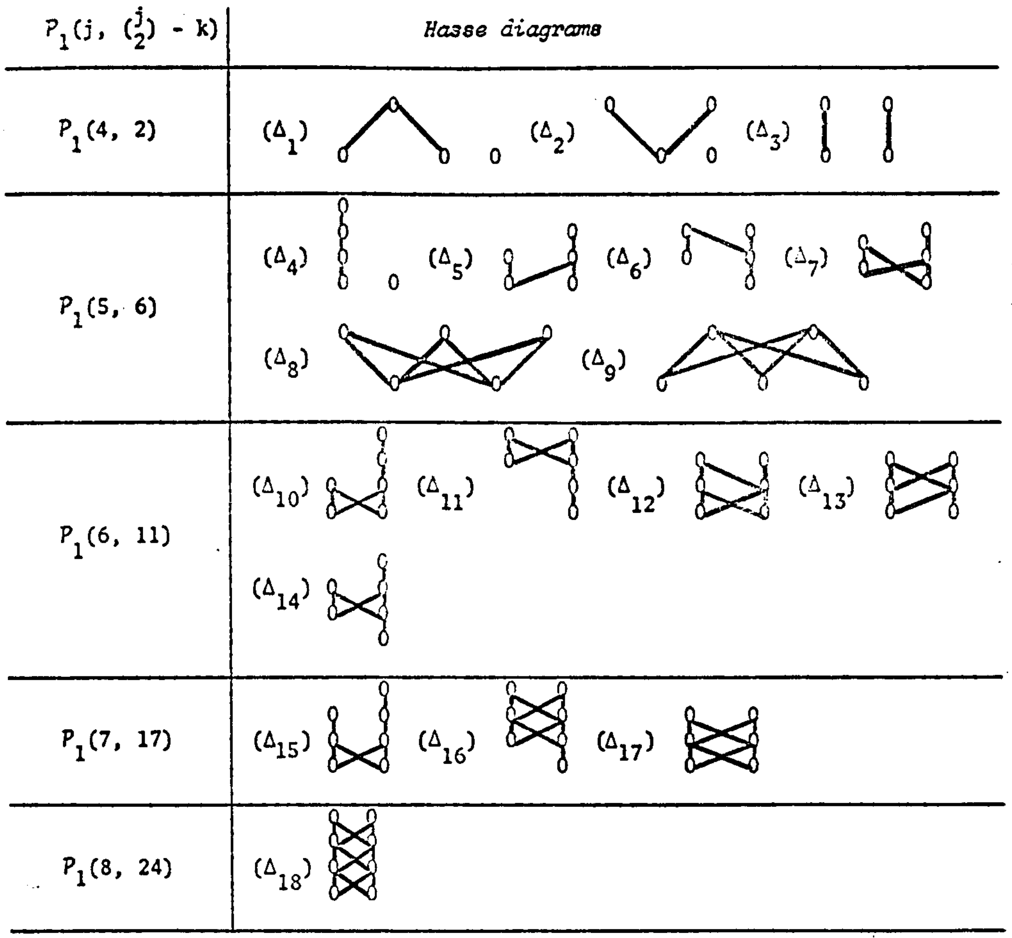


Diagram 6

Proof. For $P_1(4,2)$ and $P_1(8,24)$ the proof is straightforward. $P_1(8,24)$ is actually calculated in Corollary 1.

If $\Delta \in P_1(5,6)$, then U_Δ must have one of the following forms: (i) $\{\{a,b\}, \{a,c\}, \{a,d\}, \{a,e\}\}$; (ii) $\{\{a,b\}, \{a,c\}, \{a,d\}, \{b,e\}\}$; (iii) $\{\{a,b\} \{a,c\}, \{b,c\}, \{d,e\}\}$; (iv) $\{\{a,b\}, \{a,c\}, \{b,d\}, c,e\}\}$; (v) $\{\{a,b\}, \{a,c\}, \{b,d\}, \{d,e\}\}$.

But (v) is the same as (iv) if we relabel a as b, b as a, c as d, and d as c. Thus we only need work with cases (i)-(iv). Clearly Case (i) corresponds to Δ_4. In Case (ii) either $a > e$ or $a < e$. If $a > e$, we only get Δ_5 (up to isomorphism). Similarly, if $a < e$, we only get Δ_6. In Case (iii), either $a > d$ or $a < d$. If $a > d$, we get Δ_8, and if $a < d$, we get Δ_9. In Case (iv), either $a > d$ or $a < d$. If $a > d$, we only get

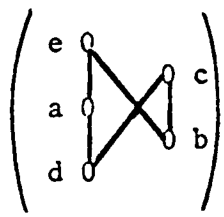
Δ_7 

Diagram 7

If $a<d$, we also get

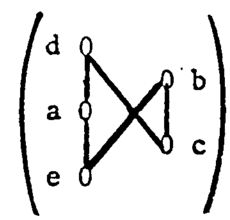
Δ_7 

Diagram 8

Thus we have calculated $P_1(5,6)$.

If $\varDelta\in P_1(6,11)$, then $U_\varDelta$ must have one of the following forms: (i) $\{\{a,b\}, \{a,c\}, \{a,d\}, \{e,f\}\}$; (ii) $\{\{a,b\}, \{a,c\}, \{b,d\}, \{e,f\}\}$; (iii) $\{\{a,b\}, \{a,c\}, \{d,e\}, \{d,f\}\}$. In Case (i), either $a>e$ or $a<e$. If $a>e$, we get $\varDelta_{10}$. If $a<e$, we get $\varDelta_{11}$. In Case (ii), either $a>d$ and $a>e$, or $a>d$, $a<e$, or $a<d$, $a>e$, or $a<d$, $a<e$. If $a>d$ and $a>e$, we get $\varDelta_{12}$. If $a>d$ and $a<e$, we get $\varDelta_{13}$. If $a<d$, $a>e$ we get $\varDelta_{12}$ again. Finally, if $a<d$, $a<e$ we get $\varDelta_{13}$ again. In Case (iii), whether $a>d$ or $a<d$ we get $\varDelta_{14}$.

If $\varDelta\in P_1(7, 17)$, $U_\varDelta$ must have the following form: $\{\{a,b\}, \{a,c\}, \{d,e\}, \{f,g\}\}$. There are only three essentially different subcases which must be considered: $a>d>f$; $d>f>a$; $d>a>f$. They give us $\varDelta_{15}, \varDelta_{16}$, and $\varDelta_{17}$ respectively.

THEOREM 6. $P(n, \binom{n}{2}-4)=(n^4-10n^3+83n^2-290n+288)/24$.

Proof. The proof is similar to the proof of Theorem 5 and is based upon Theorem 4 and Lemma 2.

REMARK. (i) It is interesting to note that the coefficients of $P(n, \binom{n}{2}-k)$, $k=0,\cdots,4$, alternate in sign. It might be interesting to investigate whether or not this is true for any k.

(ii) From Corollary 1 we know that the leading coefficient of $P(n, \binom{n}{2}-k)$ is $1/k!$. Below we shall show that the second coefficient is always $-k(k+1)/2k!$.

(iii) We now have enough information at hand to calculate

$$P(5)=\sum_{k=0}^{10} P(5, 10-k)=1+1+3+6+10+10+12+9+6+4+1=63.$$

We wish to develop a bit further the nature of the "poset polynomials" $P(n, \binom{n}{2}-k)$ along the lines used in [5] to investigate the nature of the polynomials worked with there.

DEFINITION 2. Let $C_1(j, k, q)=\{\Delta\in P_1(j, \binom{j}{2}-k): G_\Delta$ has q connected components$\}$. We let $C_1(0, 0, 0)=\{\phi\}$.

The following theorem is just a restatment of Theorem 4.

THEOREM 7. *Let n and k be as in Theorem 4, then*

$$P(n, \binom{n}{2}-k)=\sum_{j=\lambda_k}^{2k}\sum_{q=0}^{k}|C_1(j, k, q)|\binom{n-j+q}{q}.$$

From Lemmas 1 and 2 one can easily calculate the values of $|C_1(j, 3, q)|$ and $|C_1(j, 4, q)|$ for all j and q.

We will now reduce the problem of calculating $C_1(j, k, q)$ to graph theoretical notions. For the basic terminology see [4].

DEFINITON 3. (i) Let $G_1(j, k, q)=\{$undirected graphs G: G has j vertices, k edges, and q connected components with all components being nontrivial$\}$.

(ii) Let G be an undirected graph. By $\bar{G}$ we shall mean its complement.

(iii) Let G be an undirected graph. By $\tau(G)$ we shall denote the number of ways (up to isomorphism of digraphs) that an orientation can be picked for G, which turns it into a transitive digraph. Note that loops are not allowed to occur in graphs.

(iv) Let G be an undirected graph. By $\zeta(G)$ we mean a set of representatives (up to graph isomorphism) of the connected components of G. Let

$$\Psi: \zeta(G) \Longrightarrow N \text{ (N the positive integers)}$$

be given by $\Psi(A)$ is the number of connected components of G isomorphic to A, where $A\in\zeta(G)$.

THEOREM 8. *Let*

$$F: C_1(j, k, q) \Longrightarrow G_1(j, k, q)$$

be given by $F(\Delta)=G_\Delta$ (we are assuming that $C_1(j, k, q)\neq\phi$).
If $G\in G_1(j, k, q)$, then

$$|F^{-1}(G)|=q!\prod_{A\in\zeta(G)}\frac{(\tau(\bar{A}))^{\phi(A)}}{(\Psi(A))!}.$$

Proof. F is clearly well-defined. Let $G\in G_1(j, k, q)$, then there are

$$\prod_{A\in\zeta(G)}\frac{q!}{(\Psi(A))!}$$

different permutations of the connected components of G.

We form a poset $\Delta\in C_1(j, k, q)$ as follows. Suppose $B_1, \cdots, B_q$ is one of the permutations of the components of G. Pick one of the transitive orientations of $\bar{B}_i$, call it o_i. We let

$$\Delta=\bigcup_{i=1}^{q}\bar{B}_i,$$

be ordered as follows $x>y$ iff either, there exists i such that $x, y\in B_i$ and $(x, y)\in o_i$, or $x\in B_i$, $y\in B_j$, and $i<j$. It is easy to see that $\Delta\in C_1(j, k, q)$, and that $F(\Delta)=G$. Furthermore, from the discussion in Theorem 4 it is clear that we actually obtain all of $F^{-1}(G)$ in this way. Thus the theorem follows.

COROLLARY 2. *Let n and k be as in Theorem 4, then the second coefficient of $P(n, \binom{n}{2}-k)$ is $-k\ (k+1)/2k!$.*

Proof. It follows from Theorem 8 and Corollary 1, that we need only consider the terms

$$|C_1(j,k,q)|\ \binom{n-j+q}{q}$$

where $q=k-1,\ k$. Note that in orders for $C_1(j,k,q)\neq\phi$, we must have that $j\geqq 2q$. Thus we consider $C_1(2k-2,k,k-1)$, $C_1(2k-1,k,k-1)$, and $C_1(2k,k,k)$. Clearly, $C_1(2k-2,k,k-1)=\phi$, and from Corollary 1 we know that $|C_1(2k,k,k)|=1$. It is easy to see that $|G_1(2k-1,k,k-1)|=1$, and that for $G\in G_1(2k-1,\ k,k-1)$, $\zeta(G)=\{A,B\}$, where

$$A=\begin{array}{c} o\\ |\\ o\\ |\\ o\end{array}\qquad\text{and}\qquad B=\begin{array}{c} o\\ |\\ o\end{array},$$

Diagram 9

$\Psi(A)=1$, $\Psi(B)=k-2$, and $\tau(\bar{A})=\tau(\bar{B})=1$. Thus from Theorem 8 it follows that

$$|C_1(2k-1,k,k-1)|=(k-1)!/(k-2)!=k-1.$$

The coefficient of the n^{2k-1} term in $\binom{n-k}{k}$ is $-k(3k-1)/2k!$, while in $(k-1)\binom{n-k}{k-1}$ it is $(k-1)/(k-1)!$. Adding the two together we get the desired result.

We now show how some of the preceding discussion can be generalized.

DEFINITION 4. Let $m\geqq 1$. (i) Let $G_m(j,k,q)=\{$undirected graphs G: G has j vertices, k edges, q connected components, and each connected component has at least $m+1$ elements$\}$.

(ii) Let $C_m(j,k,q)=\{\Delta\in P(j,\ \binom{j}{2}-k\}: G_\Delta\in G_m(j,k,q)\}$.

Our convention is that $G_m(0,0,0)=\{\phi\}$ and $C_m(0,0,0)=\{\phi\}$

THEOREM 9. *Let*

$$F\colon C_m(j,k,q)\Longrightarrow G_m(j,k,q)$$

be given by $F(\Delta)=G_\Delta$. If $G\in G_m(j,k,q)$, then

$$|F^{-1}(G)|=q!\prod_{A\in\zeta(G)}\frac{(\tau(\bar{A}))^{\phi(A)}}{(\Psi(A))!}.$$

Proof. Identical to the proof of Theorem 8.

DEFINITION 5. Let $Gr_m=\{$undirected connected graphs G: $|G|=m\}$. Let t be an integer. By $\binom{Gr_m}{t}_r$ we mean $\{$equivalence classes (under the symmetric group on t letters) of sequences in Gr_m of length t, i.e., sets with t elements of Gr_m, repetition being allowed$\}$. If $S\in\binom{Gr_m}{t}_r$, say $S=\{B_1,\cdots,B_r\}$, by $E(S)$ we mean the set of all edges which appears in one of the B_i's (the B_i's are all assumed to be disjoint).

THEOREM 10.

$$|G_m(j,k,q)|=\sum_{\lambda=0}^{q}\ \sum_{S\in\binom{Gr_{m+1}}{\lambda}_r}|G_{m+1}(j-\lambda(m+1),\ k-|E(S)|,\ q-\lambda)|.$$

Proof. Let $\Delta\in G_m(j,k,q)$. The λ_Δ components of Δ which have exactly $(m+1)$

elements correspond to some element S_J of $\binom{Gr_{m+1}}{\lambda_J}_r$ and the $q-\lambda_J$ remaining components form an element of $G_{m+1}(j, -\lambda_J(m+1), k-|E(S)|, q-\lambda_J)$ and the theorem follows.

DEFINITION 6. Let $\Omega_m=$ {transitive digraphs up to isomorphism) D of cardinality m which arise as an orientation of a complement of some element of Gr_m}. By $\prod_{i=1}^{\lambda} \Omega_m$ we mean the Cartesian product of λ copies of Ω_m. If $T\in \prod_{i=1}^{\lambda} \Omega_m$, suppose $T=(B_1, \cdots, B_\lambda)$, by $N(T)$ we mean $\sum_{i=1}^{\lambda} E^*(B_i)$, where $E^*(B_i)$ is the number of edges in the graph G_i, the complement of which, is an undirected version of B_i, i.e., $E^*(B_i)=|B_i|-E(B_i)$.

THEOREM 11. $|C_m(j,k,q)|=\sum_{\lambda=0}^{q} \sum_{T\in\prod_{i=1}^{\lambda}\Omega_{m+1}} |C_{m+1}(j-\lambda(m+1), k-N(T), q-\lambda)|\binom{q}{\lambda}$.

Proof. The proofs of Theorem 8 and Theorem 10 contain all the essential ideas. If $\Delta\in C_m(j,k,q)$, then $G_\Delta\in G_m(j,k,q)$. Δ orders the connected components of G_Δ. Suppose there are λ_Δ connected components in G_J which have cardinality $(m+1)$. These λ_J components can occur in any of the q positions occupied by the components of G_Δ, and their removal allows us to construct an element of $C_{m+1}(j-\lambda_J(m+1), k-E, q-\lambda_J)$ along the lines of Theorem 8, where E is the number of edges in these various components. For each component A, we must take into account the $\tau(\bar{A})$ ways in which it originates. Putting everything together the theorem follows.

CONCLUDING REMARK. Theorem 11 allows us to calculate the poset polynomials while "omitting" certain information. Its numerical usefulness is bounded by the rate at which $|\Omega_m|$ grows with m. Thus we can easily make the transition to $C_2(j,k,q)$ since $|\Omega_m|=1$. Lemmas 1 and 2 can be proven using the material developed after Theorem 7.

Needless to say, the procedures outlined here work well for the cases considered in [2]. The calculations carried out in [3] can be extended to the posets considered here by making suitable modifications of the methods used there. It seems likely that several more of the poset polynomials can be calculated, but it is clear that the amount of work required increases quickly. We should note that for the actual calculation of $P(n,k)$, for some fixed n and k, using Theorem 7, it is unnecessary to calculate $|C_1(j,k,q)|$ for any $j>n$.

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Alabama State University