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The representation of posets and lattices by sets

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0. Introduction

In this paper, we present those aspects of representing posets and lattices by sets which have a combinatorial flavor. Thus the emphasis is on counting and using such concrete objects as collections of sets and binary relations.

In Section 2, we exhibit a "well-known" correspondence between residuated maps from a poset P into a poset Q and dually residuated maps from Q to P. This gives us a correspondence between sup-preserving maps from a complete lattice L_1 to a complete lattice L_2 and inf-preserving maps from L_2 to L_1. As an application of this result in Theorem 2.5 we show that the number of representations of a finite lattice L by subsets of a set of n elements such that sup corresponds to union is

$$\frac{1}{|\mathrm{Aut}\,(L)|}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(|L|-i)^n,$$

where k is the number of meet-irreducible elements of L. Among other things, this result shows that no representation is possible unless $k \leq n$.

In Section 3, we ((derive a new characterization of finite distributive lattices (Theorem 3.1): a lattice is distributive if and only if for some n it has length n, n joint-irreducible elements, n meet-irreducible elements and satisfies the Jordan-Dedekind chain condition. This characterization provides a quick and simple test for distributivity which can be applied directly to the Hasse diagram of a lattice. We also present a related characterization (Theorem 3.5) of finite locally distributive lattices due to Greene and Markowsky [11].

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In Section 4, we next show that given a sup-representation (sups are carried into unions) of a lattice L by subsets of a set X and a sup-representation of its dual L' by subsets of a set Y, then a binary relation on $X \times Y$ can be found from which both representations can be recovered in a simple manner (Theorem 4.5).

Section 5 (Theorem 5.2) presents techniques for calculating the number of order-isomorphic representations of a poset by subsets of a given set. The technique is general enough to yield the asymptotic number of representations as the size of the representing set gets large (Theorem 5.4).

Theorem 6.1 in Section 6 shows how to characterize those complete lattices which can be embedded into a given complete lattice and Theorem 6.2 relates these results to those in [16]. The results in Section 6 have application to computer science and biomathematics (see [19]) because they allow a "coordinate free" formulation of various problems.

The results in this paper complement the results described by the author in [16]. Some of these results appeared, mostly without proof, in [17].

1. Preliminaries

We will use this section to introduce some concepts and state some basic results which will be useful throughout this paper. The definitions of all terms left undefined in this paper can be found in [1].

NOTATION 1.1. Let L be a complete lattice (in particular, all finite lattices are complete.) We use 0 to denote $\inf L = \sup \emptyset$ and 1 to denote $\sup L = \inf \emptyset$. If P is a poset and $a, b \in P$, we use $[a, b]([-, b], [a, -]$ to denote the set $\{x \in P \mid a \leq x \leq b\}$ $(\{x \in P \mid x \leq b\}, \{x \in P \mid x \geq a\})$. We call $[-, b]([a, -])$ the principal (dual) ideal generated by b (a).

NOTATION 1.2. Let X and Y be sets and n an integer. $|X|$ denotes the cardinality of X, 2^X denotes the powet set of X, $\mathbf{n}$ denotes $\{1, \ldots, n\}$ (thus $\mathbf{0} = \emptyset$).

Note that $2^{X \times Y}$ is the set of *binary relations* between X and F. For $S \subset X$ $(T \subset Y)$ and $A \in 2^{X \times Y}$ we use SA (AT) to denote $\{y \in Y \mid \text{for some } x \in S, (x, y) \in A\}$ $(\{x \in X \mid \text{for some } y \in T, (x, y) \in A\})$. If $S = \{x\}$ $(T = \{y\})$ we simply write xA (Ay), omitting the brackets.

Remark. Whenever we talk about 2^X as an ordered set, we will always use the usual set inclusion ordering with infs and sups corresponding to intersections and unions.

DEFINITION 1.3. Let L be a complete lattice and $L' \subset L$. We say that L' is a sup-sublattice (inf-sublattice) if for all $X \subset L'$, $\sup_L X \in L'(\inf_L X \in L')$. Thus $0 \in L'(1 \in L')$.

Remark. It is easy to see that in both cases above, L' is a complete lattice with respect to the induced ordering. If L' is a sup-sublattice, its sup is identical with the sup in L, but its inf is simply the sup of lower bounds in L' of the set in question.

DEFINITION 1.4. Let $A \in 2^{X \times Y}$. The *row space* of A, denoted by $R(A)$, is $\{SA \mid S \subset X\}$. The *column space* of A, denoted by $C(A)$, is $\{AT \mid T \subset Y\}$.

Remark. Note that $R(A)$ $(C(A))$ is a sup-sublattice of $2^Y(2^X)$ and is thus a complete lattice.

DEFINITION 1.5. Let L and M be complete lattices, $S \subset L$, P, Q posets, $T \subset Q$ and X a set.

(a) A map $\zeta: P \to Q$ is called an *embedding* if for all $x, y \in P$, $x \leq y$ if and only if $\zeta(x) \leq \zeta(y)$. Note that ζ must be injective. If $Q = 2^X$, we say ζ is an *X-embedding.*

(b) A map $f: P \to Q$ is called *residuated* (*dually residuated*) if the inverse image of a principal (dual) ideal of Q is a principal (dual) ideal of P. Note that an injective (dually) residuated map is an embedding. Note that if P and Q are complete lattices and $f: P \to Q$ then f is residuated (dually residuated) iff f preserves arbitrary sups (infs). See [3] for further details.

(c) A map $\zeta: M \to L$ is called a *sup-embedding* (*inf-embedding*) if ζ preserves arbitrary sups(infs) and is injective. Note that in this case ζ is also an embedding, since it is an injective (dually) residuated map. If $L = 2^X$, we call ζ an *X-sup-embedding* (*X-inf-embedding*).

(d) T is a *representation* of P if T is order isomorphic to P. If $Q = 2^X$, we refer to it as an *X-representation.*

(e) S is a *sup-representation* (an *inf-representation*) of M if S is a sup-sublattice (inf-sublattice) of L and S is isomorphic to M. If $L = 2^X$, we call S an *X-sup-representation* (*X-inf-representation*).

(f) T *sup-spans* (*inf-spans*) Q if for all $w \in Q$, there exists $T_w \subset T$ such that $w = \sup T_w$ $(w = \inf T_w)$.

(g) By the *sup-rank* (*inf-rank*) of Q, we mean the smallest cardinality for which a sup-spanning (inf-spanning) set of that cardinality exists. This notion is well-defined since Q sup-spans (inf-spans) itself.

(h) By the *distributive lattice generated by P*, $D(P)$, we mean the set $\{T \subset P \mid \text{for all } x \in T, y \in P, x \geq y \text{ implies } y \in T\}$. Note that $\emptyset \in D(P)$. Sups in $D(P)$ are simply unions and infs are intersections. Note that there is a natural embedding, i, of P into $D(P)$ given by $i(x)=\{y\in P \mid y\le x\}$.

(i) $x\in Q$ is *sup-irreducible* (*inf-irreducible*) if whenever $x=\sup T$ ($x=\inf T$) for some $T\subset Q$, $x\in T$.

Remarks. It is easy to see that the image of a sup-embedding (inf-embedding) is a sup-representation (inf-representation).

We also note that any sup-spanning (inf-spanning) set must contain all sup-irreducible (inf-irreducible) elements. Furthermore, for lattices of finite length, sup-irreducible (inf-irreducible) elements are simply the join-irreducible (meet-irreducible) elements. Thus for lattices of finite length, the sup-rank (inf-rank) is simply the cardinality of the set of all sup-irreducibles (inf-irreducibles).

We conclude this section by describing the relationship between a poset P and $D(P)$.

THEOREM 1.6. *Let P be a poset and $i:P\to D(P)$ the map described in Definition* 1.5(h) *and let $S=i(P)$. Furthermore, let $g:P\to D(P)$ be given by $g(x)=\{y\in P \mid y\not\ge x\}$ and $T=g(P)$.*

(a) *For all $i(x)\in S$ ($g(x)\in T$) and $X\subset D(P)$, $i(x)\le\sup X$ ($g(x)\ge\inf X$) implies that there exists $\Delta\in X$ such that $i(x)\le\Delta$ ($g(x)\ge\Delta$).*

(b) *$S(T)$ is exactly the set of all sup-irreducibles (inf-irreducibles) of $D(P)$.*

(c) *The maps i and g are embeddings of P into $D(P)$.*

(d) *Let L be any complete lattice and $f:S\to L$ ($f:T\to L$) any isotone* (i.e., *order-preserving) map. Then there exists a* unique *sup-preserving* (*inf-preserving*) *map $\bar f:D(P)\to L$ such that $\bar f\mid S=f$ ($\bar f\mid T=f$).*

Proof. We will just sketch the proofs for S. The proofs for T are dual.

(a) From the hypothesis, we have that $x\in i(x)\le\sup X=\bigcup_{\Delta\in X}\Delta$. Thus $x\in\Delta_0$. Since $\Delta_0\in D(P)$, $i(x)\le\Delta_0$.

(b) (a) implies that everything in S is sup-irreducible. For all $Y\in D(P)$, $Y=\bigcup_{x\in Y} i(x)$, so that any element not in S is sup-reducible. Note that, for all $Y\in D(P)$, $Y=\bigcap_{x\notin Y} g(x)$.

(c) Trivial.

(d) Uniqueness follows from (b) since if $\bar f$ is sup-preserving, $\bar f(Y)=\sup\{f(i(x)) \mid x\in Y\}$. It is straightforward to verify that $\bar f$ is indeed sup-preserving.

2. The basic duality results and applications

NOTATION 2.1. Let P and Q be posets. We will use the following notation: Res (P,Q) will denote the set of all residuated maps from P to Q; ResI (P,Q) the

injective elements of Res (P,Q); ResS (P,Q) the surjective elements of Res (P,Q); DRes (P,Q) the set of all dually residuated maps from P to Q; DResI (P,Q) the set of all injective elements of DRes (P,Q); DResS (P,Q) the set of all surjective elements of DRes (P,Q).

Remark. We consider the above sets to be posets with respect to the pointwise ordering. In particular, if P and Q are complete lattices so are the sets above. The reader should always bear in mind that if P and Q are complete lattices Res (P,Q) and DRes (P,Q) are the lattices of all sup-preserving and inf-preserving maps respectively. The following theorem will be used as a starting point for our representation results. It is an instance of the adjoint functor theorem [14; p. 93]. Variants of it also occur in [3; p. 12], [7], [8] [10] and [22] so we will omit the proof.

THEOREM 2.2. *Let P and Q be posets. Then Γ*: Res $(P,Q)\to$ DRes (P,Q) *and Γ^**: DRes $(Q,P)\to$ Res (Q,P) *given by $\Gamma(f)(q)=\sup_P f^{-1}([-,q])$ and $\Gamma^*(g)(p)=\inf_Q g^{-1}([p,-])$ are poset antiisomorphisms and are inverses of one another.*

Furthermore, the images of ResI (P,Q) *and* ResS (P,Q) *by Γ are* DResS (Q,P) *and* DResI (Q,P) *respectively. In addition, for all $f\in$* Res (P,Q), *$g\in$* DRes (Q,P), *$\Gamma(f)\circ f\ge \mathrm{id}_P$ and $f\circ\Gamma(f)\le\mathrm{id}_Q$ from which it follows that $f\circ\Gamma(f)\circ f=f$ and $g\circ\Gamma^*(g)\circ g=g$. If f is surjective $f\circ\Gamma(f)=\mathrm{id}_Q$, while if f is injective, $\Gamma(f)\circ f=\mathrm{id}_P$.*

Remark. Note that an X-sup-embedding of a complete lattice L is just a member of ResI $(L,2^X)$ (see Definition 1.5(b)). By Theorem 2.2, we can construct each member of ResI $(L,2^X)$ if we know DResS $(2^X,L)$. However, $2^X=D(X)$ where we consider X to be the poset in which any two distinct elements are incomparable.

Theorem 1.6 implies that any element of DRes $(2^X,L)$ corresponds uniquely to an isotone map of the inf-irreducible elements of 2^X into L. For $x\in X$, let $m_x=\{y\in X \mid y\not\ge x\}=X-\{x\}$ be the inf-irreducible associated with x in Theorem 1.6. As in Theorem 1.6, the set $M=\{m_x \mid x\in X\}$ is order-isomorphic to X, where M is ordered by set inclusion. Since M has no order relations to preserve other than reflexivity, an arbitrary map of M into L is isotone.

Given $f:M\to L$, the inf-preserving map $\bar f:2^X\to L$ generated by f (Theorem 1.6(d)) is given by $f(Y)=\inf_L\{f(m_x) \mid x\notin y\}$. Clearly, if $\bar f$ is to be surjective, $f(M)$ must inf-span L. Thus if L is finite, $f(M)$ must include all the meet-irreducible elements. This discussion leads to the following result.

THEOREM 2.3. *Let L be a complete lattice and X a set. Then* ResI $(L,2^X)\ne\emptyset$ (DResI $(L,2^X)\ne\emptyset$) *if and only if $|X|$ is greater than or equal to the inf-rank (sup-rank) of L. If L is finite, the result states that* ResI $(L,2^X)\ne\emptyset$

DResI $(L, 2^X) \neq \emptyset$) *if and only if* $|X|$ *is greater than or equal to the number of meet-irreducible (join-irreducible) elements in L.*

Remark. Theorem 2.3 was also proved by Zaretskii [23] in response to a question by Campbell (see [1; p. 32, Ex. 5]) as to the smallest cardinality of X for which an X-inf-embedding exists. However, our techniques allow us to generalize Theorem 2.3 immediately to the case of arbitrary posets.

THEOREM 2.4. *Let P and Q be posets. If* ResI $(P, Q) \neq \emptyset$ (DResI $(P, Q) \neq \emptyset$), *then the inf-rank (sup-rank) of Q is greater than or equal to the inf-rank (sup-rank) of P.*

Proof. By Theorem 2.2, ResI (P, Q) is isomorphic to DResS (Q, P). Take any inf-spanning subset $X \subset Q$ and $g \in$ DResS (Q, P). We claim that $g(X)$ inf-spans P.

Since g is surjective, for all $p \in P$, we can find $q \in g^{-1}(p)$. There exists $\Delta \subset X$ such that $q = \inf \Delta$. Since g is isotone, $p \leq g(x)$ for all $x \in \Delta$. Let t be any lower bound of $g(\Delta)$ in P. Then for some $t^* \in Q$, $\Delta \subset [t^*, -] = g^{-1}([t, -])$. Whence $q \in [t^*, -]$ and $g(q) = p \geq t$. Thus $p = \inf_P g(\Delta)$. The other result is dual.

THEOREM 2.5. *Let L be a finite lattice and n a positive integer. Suppose L has j join-irreducible elements and m meet-irreducible elements.*

(a) $|\operatorname{Res}(L, 2^n)| = |\operatorname{Res}(2^n, L)| = |\operatorname{DRes}(L, 2^n)| = |\operatorname{DRes}(2^n, L)| = |L|^n$.

(b) $|\operatorname{ResI}(L, 2^n)| = |\operatorname{DResS}(2^n, L)| = \sum_{i=0}^{m} (-1)^i \binom{m}{i} (|L| - i)^n$.

(c) $|\operatorname{ResS}(2^n, L)| = |\operatorname{DresI}(L, 2^n)| = \sum_{i=1}^{j} (-1)^i \binom{j}{i} (|L| - i)^n$.

(d) $\lim_{n \to \infty} \frac{|\operatorname{ResI}(L, 2^n)|}{|\operatorname{Res}(L, 2^n)|} = \lim_{n \to \infty} \frac{|\operatorname{DResI}(L, 2^n)|}{|\operatorname{DRes}(L, 2^n)|} = 1.$

(Note: the sums in (b) and (c) above are equal to 0 if $n < m$ or $n < j$ respectively. For $m = n$ or $j = n$ the sums are both equal to $n!$.)

(e) The number of n-sup-representations (n-inf-representations) of L is $(1/|\operatorname{Aut}(L)|)\,|\operatorname{ResI}(L, 2^n)|$ $((1/|\operatorname{Aut}(L)|)\,|\operatorname{DResI}(L, 2^n)|)$, where Aut (L) is the automorphism group of L.

Proof. (a) That $|\operatorname{Res}(2^n, L)| = |\operatorname{DRes}(2^n, L)| = |L|^n$ follows from Theorem 1.6 and the remark following Theorem 2.2. The rest of the result follows from Theorem 2.2.

(b) From the remark following Theorem 2.2, it is clear the $|\operatorname{DReS}(2^n, L)|$ is equal to the number of mappings from n into L such that the m meet-irreducible elements are in the image. Thus we want to know the number of mappings from a set with n elements into a set of $|L|$ elements of which m elements have been singled out to always to be in the image. This number can be gotten by a simple application of the principle of inclusion and exclusion (see [13; especially p. 101]). The rest of the result follows from Theorem 2.2.

(c) Similar to (b).

(d) Clear from (a), (b) and (c).

(e) For each element in ResI $(L, 2^n)$ (DResI $(L, 2^n)$) there are clearly $|\operatorname{Aut}(L)|$ elements in ResI $(L, 2^n)$ DResI $(L, 2^n)$) having the same image.

Remark. Theorem 2.5 (b) and (c) was discovered in a semigroup context by Butler, Brandon and Hardy, and Markowsky [4] independently.

We conclude this section by giving a useful sup-embedding and inf-embedding for any finite lattice. These embeddings are easily seen to be "minimal" by reference to Theorem 2.3 or 2.5.

THEOREM 2.6. *Let L be a finite lattice, J its set of join-irreducibles and M its set of meet-irreducibles.*

(a) $f: L \to 2^M$ *given by* $f(a) = \{y \in M \mid y \not\geq a\}$ *is a sup-embedding of L into* 2^M.

(b) $g: L \to 2^J$ *given by* $g(a) = \{x \in J \mid x \leq a\}$ *is an inf-embedding of L into* 2^J.

Proof. We leave the details of the reader since they are straightforward. The key fact is that every element in L is a sup of join-irreducibles and inf of meet-irreducibles.

Remark. The representations in Theorem 2.6 above are related to the representation by principal dual ideas used by Birkhoff and Frink [2].

3. A new characterization of finite distributive and locally distributive lattices

The following theorem provides a simple test which can be applied to the Hasse diagram of a lattice to check for distributivity. It is well known (see 1; p. 58)] that any distributive lattice of length n satisfies the Jordan-Dedekind chain condition, has n join-irreducible elements and n meet-irreducible elements. The next theorem shows that these conditions are also sufficient for distributivity.

THEOREM 3.1. *Let L be a finite lattice. The following are equivalent.*

(1) *L is distributive and has n join-irreducible elements.*

(2) *L has n join-irreducible elements, n meet-irreducible elements, and every connected chain between* 1 *and* 0 *has length n.*

(3) *L has length n, satisfies the Jordan-Dedekind chain condition, has n join-irreducible elements and n meet-irreducible elements.*

Proof. It is easy to see that (2) and (3) are equivalent and it is well known that (1) implies (3). Hence, we need only show that (3) implies (1). By Theorem 2.6, L can be considered a sup-sublattice of 2^n. L'(the dual lattice) has exactly the same properties listed in (3) as L does. Consequently, by Theorem 2.6 and the fact that $2^X \cong 2^Y$ if X and Y have the same cardinalities, we can consider L' as a sup-sublattice of 2^n also. We wish to show that L and L' are sublattices of 2^n and hence distributive. Let $f: L \to L'$ be an anti-isomorphism of L into L' where we consider both as sup-lattices of 2^n. Let $\alpha \in 2^n$, by $\rho(\alpha)$ we mean the cardinality of α. Observe that since L and L' satisfy the Jordan-Dedekind chain condition, have length n, and are sup-sublattices of 2^n (which has length n) the height of $\alpha \in L$ or $\alpha \in L'$ is equal to $\rho(\alpha)$. We now make a number of claims from which our main result follows.

(1) $\rho(f(v)) = n - \rho(v)$ for all $v \in L$, since a connected chain from 0 to v is mapped into a connected chain from $f(v)$ to 1 and the height of $\alpha(\alpha \in L$ or $\alpha \in L') = \rho(\alpha)$.

(2) $\rho(f(y)) - \rho(f(x) \wedge_{L'} f(y)) = \rho(x) - \rho(x \cap y)$ for all $x, y \in L$.

$$\rho(f(y)) - \rho(f(x) \wedge_{L'} f(y)) = (n - \rho(y)) - (n - \rho(x \cup y))$$
$$= \rho(x \cup y) - \rho(y) = \rho(x) - \rho(x \cap y).$$

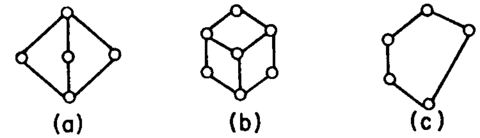
(3) $\rho(f(y)) - \rho(f(x) \cap f(y)) = \rho(x) - \rho(x \wedge_L y)$ for all $x, y \in L$.

$$\rho(f(y)) - \rho(f(x) \cap f(y)) = \rho(f(x) \cup f(y)) - \rho(f(x))$$
$$= (n - \rho(x \wedge_L y)) - (n - \rho(x)) = \rho(x) - \rho(x \wedge_L y).$$

(Recall that $\cup = \vee_L = \vee_{L'}$ since L and L' are sup-sublattices).

We know that for all $x, y \in L$, $x \wedge_L y \le x \cap y$ and $f(x) \wedge_{L'} f(y) \le f(x) \cap f(y)$. Thus $\rho(f(y)) - \rho(f(x) \cap f(y)) \le \rho(f(y)) - \rho(f(x) \wedge_{L'} f(y))$. From **(2)** and **(3)** we get $\rho(x) - \rho(x \wedge_L y) \le \rho(x) - \rho(x \cap y)$ which implies that $\rho(x \wedge_L y) \ge \rho(x \cap y)$ and hence that $x \wedge_L y = x \cap y$. Thus L is a sublattice of 2^n. Hence, both L and L' are distributive as was to be shown.

Remark. Theorem 3.1 can be rewritten as follows: a finite lattice L with n join-irreducible elements is distributive iff (i) it satisfies the Jordan-Dedekind chain condition, (ii) the number of meet-irreducible elements equals the number of join irreducible elements, (iii) the length of L is equal to the number of join-irreducible elements. The following three examples show the independence of conditions (i), (ii), and (iii).



Here $n = 3$. (a) satisfies (i) and (ii) only, (b) (i) and (iii) only, and (c) (ii) and (iii) only.

COROLLARY 3.2. *A finite modular lattice is distributive iff its length is equal to its sup-rank (inf-rank).*

Proof. It is well known that modular lattices satisfy the Jordan-Dedekind chain condition. Also Dilworth has shown [1; p. 103] that the sup-rank and inf-rank of any finite modular lattice are equal. Thus the corollary follows directly from Theorem 3.1 and these additional facts.

Greene and Markowsky [11] have shown that a finite lattice, L, satisfying (i) and (iii) is *lower locally distributive*, i.e., for each $x \in L$, if $x^* = \inf\{y \in L \mid x$ covers $y\}$, then the interval $[x^*, x]$ is a Boolean algebra. Of course, if (i) and the dual of (iii) hold for a finite lattice, it must be *upper locally distributive.* We present their proof below.

NOTATION 3.3. If L is a finite lattice we use $h(L)$ to denote the height of L, $J(L)$ to denote the set of join-irreducible elements of L and $M(L)$ to denote the set of meet-irreducible elements of L.

Let L be a finite lattice. Since every element of L is a join of join-irreducibles, it follows immediately that, if a chain length k, then its elements dominate at least k join-irreducibles. Hence we have:

LEMMA 3.4. *For any finite lattice L,*

$h(L) \le |J(L)|$. Dually, $h(L) \le |M(L)|$.

THEOREM 3.5. *If L is a finite lattice in which all maximal chains have the same length, then the following conditions are equivalent*:

(a) $h(L) = |J(L)|$;

(b) *There exists an inf-preserving, rank-preserving embedding of L into a finite distributive lattice*;

(c) *L is (lower) locally distributive.*

Proof. We give a cyclic proof that the three conditions are equivalent:

$\boldsymbol{A \Rightarrow B}$ First observe that, if $x \in L$ and $\Theta(x) = \{p \in J(L) \mid p \leq x\}$, then $|\Theta(x)| = r(x)$, where $r(x)$ is the rank of x. For $r(x) \leq |\Theta(x)|$ by Lemma 3.4, and it is easy to see that $h(L) - r(x)$ is at most the number of join-irreducibles not dominated by x. Hence $h(L) - r(x) \leq |J(L)| - |\Theta(x)|$, which together with $h(L) = |J(L)|$ implies that $r(x) = |\Theta(x)|$. Hence the injective, inf-preserving map $x \to \Theta(x)$ of Theorem 2.6 from L to the lattice of subsets of $J(L)$ is also rank-preserving.

$\boldsymbol{B \Rightarrow C}$ We may suppose that L is an inf-sublattice of a finite distributive lattice D, such that every $x \in L$ has the same rank in both L and D. If $x \in L$, the elements covered by x in L are also covered by x in D. Since D is distributive, these elements generate an interval $[x^*, x]$ in D which is a Boolean algebra, and every element of this interval is a meet of elements in L. Hence $[x^*, x]$ is an interval of L, and L is locally distributive.

$\boldsymbol{C \Rightarrow A}$ As before, we define $\Theta(x) = \{p \in J(L) \mid p \in x\}$. We will prove by induction on $r(x)$ that $|\Theta(x)| = r(x)$ for all $x \in L$. This is obviously true for elements of rank 1, so we assume that $r(x) > 1$ and also that $r(y) = |\Theta(y)|$ for all $y < x$. If x itself is join-irreducible, then x^* is the unique element convered by x, and so $|\Theta(x)| = |\Theta(x^*)| + 1 = r(x^*) + 1 = r(x)$. Otherwise, every element $p \in g_1(x)$ is less than x. If we define $C(x) = \{y \mid x \text{ covers } y\}$, then $p \in \Theta(x)$ implies $p \in \Theta(y)$ for some $y \in C(x)$. Moreover, if $p \in \Theta(y)$ for each y in some subset $A \subset C(x)$, then $p \in \Theta(\wedge_{y \in A} y)$. An easy inclusion-exclusion argument shows $|\Theta(x)| = \Sigma(-1)^{|A|-1}|\Theta(\wedge_{y \in A} y)|$ (where) the sum is over all nonempty $A \subset C(x)$). If we define $y_A = \wedge_{y \in A} y$, then local distributivity implies that $r(y_A) = r(x) - |A|$. Hence, the above sum becomes $|\Theta(x)| = \Sigma(-1)^{|A|-1}(r(x) - |A|) = r(x)$ (where the sum is over all nonempty $A \subset C(x)$) by elementary binomial manipulation. This proves that $|\Theta(x)| = r(x)$ for each $x \in L$, and hence, $|J(L)| = |\Theta(1)| = r(1) = h(L)$, where 1 is the top element of L.

Remark. Dilworth [9; Corollary 1.4] showed that a lattice which is both upper and lower locally distributive is distributive. Thus the result and Theorem 3.5 can be used to give a different proof of Theorem 3.1. Local distributivity is also discussed in [6].

4. Binary relations and lattices

In this section we will show how binary relations can be thought of as arising from a sup-embedding of a complete lattice and a sup-embedding of its dual. We will also indicate how some of the preceding results can be obtained from this viewpoint. The following theorem is a key result in this section.

THEOREM 4.1. *Let X and Y be sets and $A \in 2^{X \times Y}$. Then the map $f: C(A) \to R(A)$ given by $f(AT) = (X - AT)A$ and the map $g: R(A) \to C(A)$ given by $g(SA) = A(Y - SA)$ are anti-isomorphisms of complete lattices and are inverses of one another.*

Proof. Clearly f and g are order-inverting. We will now show that $g(f(AT)) = AT$ for all $T \subset Y$. $x \in g(f(AT)) \Leftrightarrow xA \cap (Y - f(AT)) \neq \emptyset \Leftrightarrow xA \not\subset f(AT) = (X - AT)A \Leftrightarrow x \notin X - AT \Leftrightarrow x \in AT$.

The proof that for all $S \subset X$, $f(g(SA)) = SA$, is similar. Thus f and g are order reversing bijections, which implies that they are lattice anti-isomorphisms.

Remark. The reader familiar with the concept of Galois connection [1; p. 124], might recognize that we essentially have a Galois connection here. This construction is closely related to the concept of a polarity [1; p. 122] and was used in the case $X = Y$ by Zaretski [24] in his work on the structure of the semigroup of binary relations. Variants of Theorem 4.1 can be found in [3; p. 35, exercise 4.15], [7], [8], [10], [20; Theorem 10]. We will now show how to derive part of Theorem 2.3 from Theorem 4.1.

THEOREM 4.2. *Let L be a complete lattice and X a set. If $\text{ResI}(L, 2^X) \neq \emptyset$, then $|X|$ is greater than or equal to the inf-rank of L. Of course the dual statement is also true.*

Proof. Let $f \in \text{ResI}(L, 2^X)$. Let $A_f \subset L \times X$ be given by $(a, x) \in A_f$ if and only if $x \in f(a)$. Since f is sup-preserving, $R(A_f) = f(L) \simeq L$. Note that $C(A_f) \simeq L'$ by Theorem 4.1 where L' is the dual of L. Note that the set $\{A_f y \mid y \in X\}$ sup-spans L'. From the anti-isomorphism given in Theorem 4.1 we can now construct a set of cardinality $\leq |X|$ which inf-spans L.

With the use of Theorem 2.6 we can also derive Theorem 2.4 for complete lattices L_1 and L_2.

Alternative proof of Theorem 2.4. Take a set Y of smallest cardinality which inf-spans L_2. By Theorem 2.6 we can sup-embed L_2 in 2^Y. Since compositions of sup-embeddings are sup-embeddings, L_1 can be sup-embedded in 2^Y. The result now follows from Theorem 4.2. The rest follows by duality.

The next lemma relates the results of Theorem 4.1 to sup-embeddings of a lattice L.

LEMMA 4.3. *Let $A \subset X \times Y$ and L a complete lattice isomorphic to $C(A)$ via ζ. Let L' be the lattice dual to L and let $r: L' \to L$ be any anti-isomorphism. Let*

$\zeta' : L' \to R(A)$ be the isomorphism given by $\zeta' = f\zeta r$ where f is the map of Theorem 4.1. For $x \in X$ ($y \in Y$) let $l_x = \sup_L \{l \in L \mid x \notin \zeta(l)\}$ ($k_y = \sup_{L'} \{k \in L' \mid y \notin \zeta'(k)\}$). Then for all $x \in X$ ($y \in Y$) $xA = \zeta'(r^{-1}(l_x)) = f\zeta(l_x)(Ay = \zeta r(k_y))$.

Proof. Let g be as in Theorem 4.1. Since f and g are inverses, $fg(xA) = xA$. However, $g(xA) = A(Y - xA)$. We claim that $A(Y - xA) = \zeta(l_x)$.

Note that $x \notin A(Y - xA)$. Thus $A(Y - xA) \leq \zeta(l_k)$.

Now $\zeta(l_x) \in C(A)$, i.e., $\zeta(l_x) = AT$ for some $T \subset Y$. Note that $T \cap xA = \varnothing$ else $x \in AT = \zeta(l_x)$. Since ζ is sup-preserving, $x \notin \zeta(l_x)$, whence $T \subset Y - xA$, which implies that $\zeta(l_k) \leq A(Y - xA)$.

Arguing as above, we see that for all $y \in Y$, $f(Ay) = \zeta(k_y)$, thus $Ay = gf(Ay) = g'(k_y) = gfr(k_y) = r(k_y)$.

LEMMA 4.4. *Let $\zeta : L \to 2^X$ be an X-sup-embedding of the complete lattice L and $\zeta' : L' \to 2^Y$ a Y-sup-embedding of its dual. Let $r : L' \to L$ be any anti-isomorphism and $A \subset X \times Y$ be given by $(x, y) \in A$ if and only if $y \in \zeta' r^{-1}(l_x)$ (or equivalently $x \in \zeta r(k_y)$). Then $C(A) = \zeta(L)$ and $R(A) = \zeta'(L')$.*

Proof. We claim that $(x, y) \in A$ if and only if $x \in \zeta r(k_y)$. This follows from the fact that $x \in \zeta r(k_y) \Leftrightarrow r(k_y) \leq l_x \Leftrightarrow k_y \geq r^{-1}(l_x) \Leftrightarrow y \notin \zeta' r^{-1}(l_x)$.

Observe that for all $x \in X$ ($y \in Y$) $xA = \zeta'(r^{-1}(l_x)) \in \zeta'(L')((Ay - \zeta r(k_y) \in \zeta(L))$. Thus $R(A) \subset \zeta'(L')$ and $C(A) \subset \zeta(L)$.

Let $l_0 \in L$, we will show that $\zeta(l_0) = A(Y - \zeta' r^{-1}(l_0))$. If $x \in A(Y - \zeta' r^{-1}(l_0))$, then $x \in Ay = \zeta r(k_y)$ for some $y \notin \zeta' r^{-1}(l_0)$. But $y \notin \zeta' r^{-1}(l_0) \Rightarrow r^{-1}(l_0) \leq k_y \Rightarrow l_0 \geq r(k_y) \Rightarrow \zeta(l_0) \geq \zeta r(k_y) \Rightarrow x \in \zeta(l_0)$. Let $x \in \zeta(l_0)$ and recall that $x \notin \zeta(l_x)$. Thus $l_x \not\geq l_0 \Rightarrow r^{-1}(l_x) \not\leq r^{-1}(l_0)$. Let $y \in \zeta' r^{-1}(l_x) - \zeta' r^{-1}(l_0)$. Then $(x, y) \in A$ and $y \in Y - \zeta' r^{-1}(l_0)$. Thus $x \in A(Y - \zeta' r^{-1}(l_0))$. Thus $C(A) \supset \zeta(L)$. Similarly, $R(A) \supset \zeta'(L')$.

Remark. Lemma 4.4 shows that we can "glue" together a sup-representation of L and one of its dual to get a binary relation from which both representations can be recovered. The reader will note that the "glueing" process involves a choice of an anti-isomorphism between L and L'. Thus there would seem to be at least $|\operatorname{Aut}(L)|$ binary relations which could be obtained. The next theorem pursues this point further.

THEOREM 4.5. *Let ζ and Θ be X-sup-embeddings of a complete lattice L and ζ' a Y-sup-embedding of its dual L'. Let $A \subset X \times Y$ be such that $C(A) = \zeta(L)$ and $R(A) = \zeta'(L')$ (A exists by Lemma 4.4) and let $\Omega = \{B \subset X \times Y \mid C(B) = \Theta(L)$ and $R(B) = \zeta'(L')\}$. Then the map $F_A : \operatorname{Aut}(\zeta(L)) \to \Omega$, given by $(x, y) \in F_A(f)$ if and only if $x \in \Theta f \zeta^{-1}(Ay)$, is a bijection.*

Of course, a dual result holds where we have two embeddings of L' and one of L and we change $R(A)$.

Proof. Let $r : L' \to L$ be any anti-isomorphism. By Lemma 4.3, $Ay = \zeta r(k_y)$ for all $y \in Y$. Thus $(x, y) \in F_A(f)$ if and only if $x \in \Theta fr(k_y)$. Note that $fr : L' \to L$ is an anti-isomorphism. By Lemma 4.4, $C(F_A(f)) = \Theta(L)$ and $R(F_A(f)) = \zeta'(L')$. Thus F_A is well defined.

F_A is injective, since for all $y \in Y$, $F_A(f)y = \Theta f \zeta^{-1}(Ay)$. If $F_A(f) = F_A(g)$, then for all y we would have $g^{-1} f \zeta^{-1}(Ay) = \zeta^{-1}(Ay)$. Since $\{\zeta^{-1}(Ay) \mid y \in Y\}$ sup-spans L, $g^{-1}f$ must be the identity.

It remains to show that F_A is surjective. Let $B \in \Omega$ and $\delta : L \to \Theta(L)$ be given by $\delta(l) = \sup_{\Theta(L)} \{By \mid \zeta^{-1}(Ay) \leq l\}$.

We first show that for all Δ, Δ', $\cup_{y \in \Delta} Ay \subset \cup_{y \in \Delta'} Ay$ if and only if $\cup_{y \in \Delta} By \subset \cup_{y \in \Delta'} By$. Suppose $\cup_{y \in \Delta} Ay \subset \cup_{y \in \Delta'} Ay$, but $\cup_{y \in \Delta} By \not\subset \cup_{y \in \Delta'} By$. Then there exists $y_0 \in \Delta$ and $x_0 \in By_0$ such that for all $y \in \Delta'$, $x_0 \notin By$. Thus $y_0 \in x_0 B$ and $x_0 B \cap \Delta' = \emptyset$. Since $x_0 B \in R(B) = R(A)$, there exists $x_1 \in X$ such that $y_0 \in x_1 A \subset x_0 B$. Then we have $x_1 \in Ay_0 \subset \cup_{y \in \Delta} Ay$, but $x_1 \notin \cup_{y \in \Delta'} Ay$, since $x_1 A \cap \Delta' \subset x_0 B \cap \Delta' = \emptyset$. Clearly, the same argument works in the other direction.

It now follows that for all $l, l^* \in L$, $l \leq l^*$ if and only if $\delta(l) \leq \delta(l^*)$. Thus δ is a lattice isomorphism between L and $\Theta(L)$ and is thus an X-sup-embedding of L. Let $f = \zeta \Theta^{-1} \delta \in \operatorname{Aut}(\zeta(L))$. Clearly, $F_A(f) = B$.

Remark. In the case $X = Y$, the results in this section imply many things about the structure of the semigroup of binary relations. The last result (when $\Theta = \zeta$) can be extended to a dual group isomorphism between $\operatorname{Aut}(L)$ and the Schutzenberger group of the H-class of A. For additional details and applications, see [5, 18, 24].

One of the referees noted that the material in Lemmas 4.3, 4.4 and Theorem 4.5 could be derived and put in a more general context using the correspondence between binary relations on $X \times Y$ and $\operatorname{Res}(2^X, 2^Y)$ and the material in [3; especially p. 36, exercise 4.17]. We will not pursue this point further.

The numerical results of Theorem 2.5 can be derived from Theorem 4.5. However, we shall omit this derivation.

5. Embeddings of posets into complete lattices

In this section, we shall be concerned with the question of the number of embeddings of a poset in a complete lattice. We will be especially interested in the number of embeddings of a finite poset in 2^n for n a positive integer. Abraham

Hillman [12] studied this problem for posets with 4 or fewer elements. We believe that the technique presented here gives greater insight into the problem of representing posets than Hillman's techniques. We will study the problem of embedding one lattice into another from a different point of view in the next section.

The first theorem characterizes poset embeddings in terms of lattice sup-embeddings. This is a fairly natural approach since we have already developed a fair amount of material in this case. The following definition is a key step in the transition.

DEFINITION 5.1. Let L be a complete lattice and $S \subset L$. Naturally, we think of S as a poset with respect to the induced order. By the *sup-sublattice generated by* S, L_S, we mean $\{\sup_L T \mid T \subset S\}$.

THEOREM 5.2. *Let L be a complete lattice and $S \subset L$. Let $f: L_S \to D(S)$ be given by $f(a) = \{x \in S \mid x \le a\}$.*

(a) $f \in \mathrm{DResI}(L_S, D(S))$. *Thus $f(L_S)$ is an inf-sublattice of $D(S)$ isomorphic to L_S.*

(b) *Let i be the map of Definition* 1.5(h). *Then $i(S) \subset f(L_S)$ and $i(S)$ sup-spans $f(L_S)$.*

Proof. (a) Clearly f is well-defined, isotone and injective. Let $A \subset L_S$ and $a = \inf_{L_S} A$. Then for all $x \in S$, $x \le a$ if and only if $x \le b$ for all $b \in A$, since $a = \sup_L \{x \in S \mid x \le b$ for all $b \in B\}$. Thus $f(a) = \bigcap_{b \in A} f(b) = \inf_{D(S)} f(A)$.

(b) For each $x \in S$, $f(x) = i(x)$. Since $i(S)$ sup-spans $D(S)$, it certainly sup-spans $f(L_S)$.

Remark. The point of Theorem 5.2 is that in order to get all the representations of a poset, P, we need to know the inf-sublattices of $D(P)$ which contains $i(P)$. Finding these inf-sublattices may be difficult in general. However, the theorem implies that all representations of a poset can be classified depending on which type of L_S is generated. Thus, there are a fixed finite number of "types" of possible representations of a finite poset regardless of the lattice into which we wish to embed the poset.

While finding all the inf-sublattices of $D(P)$ may be hard in general, the technique described above will enable us to calculate the asymptotic number of representations of a poset by subsets of $\mathbf{n}$ as $n \to \infty$. Before proceeding to this and similar results, we shall first introduce some additional notation.

NOTATION 5.3. Let L and M be complete lattices and P a poset. We will use **RS(M, L)** to denote the set of all sup-representations of M in L. We use **R(P, L)** to denote all representations of P in L and $\mathbf{R^*(P, L)}$ to denote all representations of P in L which sup-span L. If $L = 2^{\mathbf{n}}$, we simply write **RS(M, n)**, **R(P, n)** and $\mathbf{R^*(P, n)}$.

We use **Z(P)** to denote the set of all inf-sublattices of $D(P)$ which contain $i(P)$. Finally, we let **ZE(P)** be a set which contains exactly one element from each lattice-isomorphism class of $Z(P)$.

Remark. Note that $|\mathrm{ResI}(M, L)| = |RS(M, L) \times \mathrm{Aut}(M)|$. Note also that Theorem 2.5(e) allows us to calculate $RS(M, n)$.

THEOREM 5.4. *Let P be a poset and L a complete lattice.*

(a) $|R(P, L)| = \sum_{M \in ZE(P)} |RS(M, L) \times R^*(P, M)|$.

(b) $|\mathrm{ResI}(D(P), L)| \le |R(P, L) \times \mathrm{Aut}(P)| \le |\mathrm{Res}(D(P), L)|$.

(c) If P is finite, then $\lim_{n \to \infty} \dfrac{|R(P, n)| x |\mathrm{Aut}(P)|}{|D(P)|^n} = 1$.

Proof. (a) The easiest way to prove this is to construct a bijection between the sets. Let $\Gamma: \bigcup_{M \in ZE(P)} RS(M, L) \times R^*(P, M) \to R(P, L)$. For each $M \in ZE(P)$ and $N \in RS(M, L)$ pick an isomorphism $h_{M,N}: M \to N \subset L$. For $N \in RS(M, L)$ and $\Delta \in R^*(P, M)$, let $\Gamma(N, \Delta) = h_{M,N}(\Delta)$. It follows from Theorem 5.2 that Γ is a surjection. Suppose $A \underset{\text{def}}{=} h_{M_1,N_1}(\Delta_1) = h_{M_2,N_2}(\Delta_2) \underset{\text{def}}{=} B$. Then $N_1 = L_A = L_B = N_2$ and $M_1 \simeq M_2 \in ZE(P)$. Thus $M_1 = M_2$. Since $h_{M_1,N_1}(\Delta_1) = h_{M_1,N_1}(\Delta_2)$, and h_{M_1,N_1} is a bijection, $\Delta_1 = \Delta_2$.

(b) It is easy to see that $R(P, L) \times \mathrm{Aut}(P)$ corresponds in a natural way to the set of all embeddings of P into L. Since any $f \in \mathrm{ResI}(D(P), L)$ gives rise to the embedding fi of P into L, and these embeddings are all distinct, $|\mathrm{ResI}(D(P), L)| \le |R(P, L) \times \mathrm{Aut}(P)|$.

To get the other inequality we observe that the set of all embeddings of P into L is a subset of all isotone maps from P to L. By Theorem 1.6(d) this last set has the same cardinality as $\mathrm{Res}(D(P), L)$.

(c) This follows from (b) and Theorem 2.5.

Remark. We will now consider the representations of specific families of posets. Abraham Hillman [12] calculated $R(P, n)$ for all P such that $|P| \le 4$.

His work can be duplicated using our basic approach (see [15]). However, we will concentrate on those cases which will best display our techniques.

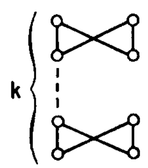
NOTATION 5.5. Let k be a positive integer.

(a) Let C_k denote a k-element chain.

(b) Let A_k denote the k-element totally unordered poset, i.e., $a \le b$ if and only if $a = b$.

(c) Let $D_k = \mathbf{k} \times \mathbf{2}$, ordered as follows $(k_1, a_1) \ge (k_2, a_2)$ if and only if $(k_1, a_1) = (k_2, a_2)$ or $k_1 > k_2$.

Remark. It should be clear what the Hasse diagrams of A_k and C_k look like. Below is the Hasse diagram of D_K.



THEOREM 5.6. *Let k be a positive integer.*

(a) $|R(C_k, n)| = \sum_{j=0,k} (-1)^j \binom{k}{j}(k-j+1)^n + \sum_{j=0,k-1} (-1)^j \binom{k-1}{j}(k-j)^n$.

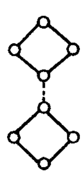
(b) $|R(D_k, n)| = 2^{-k}\left(\sum_{j=0,2k} (-1)^j \binom{2k}{j}(3k+1-j)^n\right)$.

(c) As $n \to \infty$, $|R(A_k, n)| \sim \frac{2^{kn}}{k!} \sim \binom{2^n}{k}$, i.e.,

the ratio of any two of the quantities goes to 1 *as* $n \to \infty$.

Proof. (a) $D(C_k) = C_{k+1}$. It is easy to see that $ZE(C_k) = \{C_k, C_{k+1}\}$. The result now follows from Theorem 2.5.

(b) $D(D_k)$ looks like



A little reflections shows that $ZE(P) = \{D(D_k)\}$. The result now follows from Theorem 2.5.

(c) $D(A_k) = 2^{\mathbf{k}}$. The result now follows from Theorem 5.4.

Remark. Theorem 5.6 gives some idea of the families of posets for which it is comparatively simple to calculate the number of representations. We note that (a) of Theorem 5.6 was calculated by Hillman [12].

$|R(A_k, n)|$ is the number of anti-chains of size k in $\mathbf{2}^{\mathbf{n}}$. Thus Theorem 5.6(c) asserts that for n large and k fixed almost every k element subset of $\mathbf{2}^{\mathbf{n}}$ is an anti-chain. Calculating the quantity $|R(A_k, n)|$ is difficult because of the large number of elements in $ZE(A_k)$. $ZE(A_4)$ has more than 37 elements. Hillman [12] has calculated $|R(A_k, n)|$ for $k \in \mathbf{4}$ and Riviere [21] has calculated $|R(A_k, n)|$ for $k \in \mathbf{3}$. However, neither one makes any statement about the asymptotic behavior of $|R(A_k, n)|$.

The following obvious theorem has some interesting implications which we will briefly discuss after the theorem.

THEOREM 5.7. *Let P be a poset and P' its dual and L be a complete, self-dual lattice. Then, $|R(P, L)| = |R(P', L)|$. In particular, $|R(P, n)| = |R(P', n)|$.*

Remark. The interesting thing is that if one tries to prove Theorem 5.7 starting from Theorem 5.4, one runs into difficulties right from the start. The reason being that the cardinalities of $ZE(P)$ and $ZE(P')$ are in general different and there is no obvious relationship between the lattices occurring there.

Let P be the poset represented by  and P' its dual. One can quickly calculate that $ZE(P)$ has exactly 10 elements and for each element $|R^*(P, M)| = 1$. However, $ZE(P')$ has 9 elements, 7 with $|R^*(P, M)| = 1$, 1 with $|R^*(P, M)| = 2$ and 1 with $|R^*(P, M)| = 3$. Investigating the exact nature of the relationship between $ZE(P)$ and $ZE(P')$ might prove quite interesting.

6. Embeddings of complete lattices into complete lattices

In this section we present a simply necessary and sufficient condition for one complete lattice to be capable of being embedded into another. We then interpret this result in terms of binary relations. This last interpretation connects naturally with the bidigraph representation of a lattice by a sup-spanning subset and an inf-spanning subset introduced by the author in [15, 16].

THEOREM 6.1. *Let L and P be complete lattices, X, Y subsets of L and A, B*

subsets of P such that $X(Y)$ sup-spans (inf-spans) L and $A(B)$ sup-spans (inf-spans) P. Then the following are equivalent.

(i) *There exists an embedding $\Theta : P \to L$.*

(ii) *There exists maps $f: A \to 2^X$ and $g: B \to 2^Y$ such that for all $a \in A$, $b \in B$, $a \not\leq b$ iff there exists $x \in f(a)$, $y \in g(b)$ such that $x \not\leq y$.*

Proof. **(i) ⇒ (ii)** Let f be given by $f(a) = \{x \in X \mid x \leq \Theta(a)\}$ and g by $g(b) = Y - \{y \in Y \mid y \not\geq \Theta(a)$ for some $a \in A$ such that $a \leq b\}$.

Suppose there exist $x \in f(a)$, $y \in g(b)$ such that $x \not\leq y$. Since $y \in g(b)$, $y \geq \Theta(\bar{a})$ for all $\bar{a} \in A$ such that $\bar{a} \leq b$. However, $x \in f(a)$ implies $x \leq \Theta(a)$ and since $x \not\leq y$, $\Theta(a) \not\leq y$. Thus $a \not\leq b$.

Conversely, if $a \not\leq b$, $\Theta(a) \not\leq \Theta(b)$. Since X sup-spans L and Y inf-spans L, there exist $x \in X$, $y \in Y$ such that $x \leq \Theta(a)$, $\Theta(b) \leq y$ and $x \not\leq y$. Clearly, $x \in f(a)$ and $y \in g(b)$.

(ii) ⇒ (i) Let Θ be given by $\Theta(p) = \sup \bigcup_{a \leq p} f(a)$. If $p \leq q$, then clearly $\Theta(p) \leq \Theta(q)$.

Suppose $p \not\leq q$, then there exist $a \in A$, $b \in B$ such that $a \leq p$, $q \leq b$ and $a \not\leq b$. Thus there exist $x \in f(a)$, $y \in g(b)$ for which $x \not\leq y$. Note that $x \leq \Theta(p) \not\leq y$. We claim that $y \geq \Theta(q)$. If not, for some $\bar{a} \in A$ such that $\bar{a} \leq q$, $y \not\geq \sup f(\bar{a})$. Thus there exists $x' \in f(\bar{a})$ for which $x' \not\leq y$. By (ii) we would have $\bar{a} \not\leq b$ which is impossible since $\bar{a} \leq q \leq b$. Since $y \geq \Theta(q)$ and $y \not\geq \Theta(p)$, $\Theta(q) \not\geq \Theta(p)$ and Θ is an embedding.

THEOREM 6.2. *Let X, Y, A and B be sets, $\Omega \subset X \times Y$, $M \subset A \times B$. The following are equivalent.*

(i) *There exists an embedding $\Theta : R(M) \to R(\Omega)$.*

(ii) *There exists maps $f: A \to 2^X$ and $g: B \to 2^Y$ such that for all $a \in A$, $b \in B$, aMb if and only if $f(a)\Omega \cap g(b) \neq \emptyset$.*

Proof. **(i) ⇒ (ii)** For $a \in A$, let $f(a) = \{x \in X \mid x\Omega \leq \Theta(aM)\}$. Thus $f(a)\Omega = \Theta(aM)$. For $b \in B$, let $g(b) = Y - \bigcup_{d \in Mb} \Theta(dM)$. Clearly if $f(a)\Omega \cap g(b) = \Theta(aM) \cap g(b) \neq \emptyset$, aMb.

Suppose aMb, but $\Theta(aM) \cap g(b) = \emptyset$. Then $\Theta(aM) \leq \bigcup_{d \in Mb} \Theta(dM) \leq \Theta(SM)$, where $S = A - Mb$, since $SM \geq dM$ for all $d \in S$ and since Θ is isotone. Since Θ is an embedding, $aM \leq SM$ which is impossible since aMb but $b \notin SM$. Thus $f(a)\Omega \cap g(b) \neq \emptyset$.

(ii) ⇒ (i) For $S \subset A$, let $\Theta(SM) = \bigcup_{a \in \bar{S}} f(a)\Omega$ where $\bar{S} = \{a \in A \mid aM \leq SM\}$. If $SM \leq TM$, $\bar{S} \leq \bar{T}$ and $\Theta(SM) \leq \Theta(TM)$.

Suppose $\Theta(SM) \leq \Theta(TM)$. If $b \in SM$, then $\Theta(SM) \cap g(b) \neq \emptyset$ and thus $\Theta(TM) \cap g(b) \neq \emptyset$. Hence for some $a \in \bar{T}$, $b \in aM \leq TM$. Thus $SM \leq TM$.

Remark. The connection between Theorems 6.1 and 6.2 is exactly the notion of $Q(X, Y, L)$ in [16]. Theorem 6.1 in particular shows that P can be embedded into L if and only the bidigraph $Q(A, B, P)$ can be embedded as a bidigraph into $Q(L - \{0\}, L - \{1\}, L)$. Since we don't want to introduce all the results of [16], we leave the verification of this fact to the reader.

Relationships between the embedding theorems in this section and problems in biomathematics and computer science are explored in [19].

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