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THE SCHÜTZENBERGER GROUP OF AN $\mathcal{H}$-CLASS IN THE SEMIGROUP OF BINARY RELATIONS

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K. A. Zaretskii has associated a lattice $V(\alpha)$ with each binary relation α, and he has shown that H_α is isomorphic with the group of all automorphisms of $V(\alpha)$ if H_α is a group. This result is extended in this paper by showing that for any binary relation α, the Schützenberger group $\Gamma(H_\alpha)$ is isomorphic with the group of all automorphisms of $V(\alpha)$.

It is well known that a group can be associated with any element x of any semigroup S by taking the Schützenberger group of the $\mathcal{H}$-class containing x [3, Theorem 2.22]. In this paper, a second group is associated with each element of the semigroup $\mathcal{B}_X$ of binary relations on the set X. For certain relations a third group is also studied. Each of these groups is shown to be isomorphic to the Schützenberger group. Thus, the cardinality of an $\mathcal{H}$-class H_α is equal to the cardinality of each of the three groups associated with α.

Formulas have been given for the cardinalities of the $\mathcal{L}$-, $\mathcal{R}$- and $\mathcal{D}$-classes containing the relation α if X is finite [2]. The formula for the cardinality of the $\mathcal{D}$-class D_α depends upon the cardinality of the $\mathcal{H}$-class H_α. To compute the cardinality of H_α it is sufficient to compute the cardinality of any of the three groups associated with α. Starting with the definition, it is a difficult task to compute the cardinality of a Schützenberger group. The second group is a group of automorphisms of a certain lattice, and the structure of this group is sometimes obvious by inspection of the lattice. The third group is a subgroup of the symmetric group on X.

DEFINITIONS

Let $\mathcal{B}_X$ be the set of all binary relations on the set X. For α and $\beta \in \mathcal{B}_X$, define a product $\alpha\beta$ by $(x,y) \in \alpha\beta$ if and only if there exists $u \in X$ such that $(x,u) \in \alpha$ and $(u,y) \in \beta$. This makes $\mathcal{B}_X$ into a semigroup, called the <u>semigroup of binary relations on</u> X.

Let $\alpha \in \mathcal{B}_X$, $x \in X$, and $A \subseteq X$. Then

$$\begin{aligned} x\alpha &= \{y : (x,y) \in \alpha\}, \\ A\alpha &= \{y : (x,y) \in \alpha \text{ for some } x \in A\} \\ &\quad (= \cup_{x\in A} x\alpha), \\ \alpha^{-1} &= \{(x,y) : (y,x) \in \alpha\}, \\ A' &= \{x \in X : x \notin A\}, \text{ and} \\ V(\alpha) &= \{A\alpha : A \subseteq X\}. \end{aligned}$$

The set $x\alpha$ is called a <u>row</u> of α and the set $x\alpha^{-1}$ is called a <u>column</u> of α (or a row of α^{-1}). The set $V(\alpha)$ is a lattice under set inclusion, and is called the <u>row lattice</u> of α. The lattice $V(\alpha^{-1})$ is the <u>column lattice</u> of α. A relation α is called <u>row reduced</u> if for each $x \in X$, either (i) $x\alpha = \square$ or (ii) $A \subsetneq X$ and $x\alpha = A\alpha$ implies $x \in A$. It follows that α is row reduced if no non-empty row of α is a union of other rows of α. A relation is <u>reduced</u> if both α and α^{-1} are row reduced [5].

Let $(L,\vee,\wedge)$ and $(L',\vee,\wedge)$ be lattices. A <u>lattice isomorphism</u> is a bijection $\phi : L \to L'$ such that $(A \vee B)\phi = (A\phi) \vee (B\phi)$ and $(A \wedge B)\phi = (A\phi) \wedge (B\phi)$ for all $A,B \in L$. By Birkhoff [1, p. 22], ϕ is a lattice isomorphism if and only if $\phi : L \to L'$ is a bijection such that $A \leq B$ if and only if $A\phi \leq B\phi$. A <u>lattice automorphism</u> is an isomorphism $\phi : L \to L$. A <u>lattice anti-isomorphism</u> is a bijection $\psi : L \to L'$ such that $A \leq B$ if and only if $A\psi \geq B\psi$. Let Aut(L) denote the group of all lattice automorphisms of L.

Let $\Gamma(H)$ be the Schützenberger group of the $\mathcal{H}$-class H; that is, $\Gamma(H)$ is the set of all inner right translations ρ_α restricted to H such that $H\alpha \subseteq H$. All other notation follows that of Clifford and Preston [3].

PRELIMINARY RESULTS

The next two lemmas follow directly from definitions.

<u>LEMMA 1</u>. <u>Let α be any relation in $\mathcal{B}_X$ and let A and B be subsets of X. Then $A \subseteq B$ implies $A\alpha \subseteq B\alpha$.</u>

<u>LEMMA 2</u>. <u>If α and β are in $\mathcal{B}_X$, then $\alpha\mathcal{L}\beta$ if and only if $\alpha^{-1}\mathcal{R}\beta^{-1}$.</u>

<u>LEMMA 3</u>. <u>If $\alpha \in \mathcal{B}_X$ and $A,B \subseteq X$, then $A\alpha \cap B = \square$ if and only if $A \cap B\alpha^{-1} = \square$.</u>

Proof. Both statements are equivalent to the non-existence of $a \in A$ and $b \in B$ such that $(a,b) \in \alpha$.

The following lemma is due to Zaretskii [6]. A proof is given for completeness.

<u>LEMMA 4</u>. <u>If $A \subseteq X$ and $\alpha \in \mathcal{B}_X$, then $((((A\alpha)')\alpha^{-1})')\alpha = A\alpha$.</u>

Proof. Since $A\alpha \cap (A\alpha)' = \square$, it follows from Lemma 3 that $A \cap ((A\alpha)')\alpha^{-1} = \square$, and hence $A \subseteq (((A\alpha)')\alpha^{-1})'$. By Lemma 1, $A\alpha \subseteq ((((A\alpha)')\alpha^{-1})')\alpha$.

To show that reverse inclusion, we note that for any subset B of X, $x \in B\alpha$ implies $x\alpha^{-1} \cap B \neq \square$, so that $x\alpha^{-1} \not\subseteq B'$. In particular, if $x \in ((((A\alpha)')\alpha^{-1})')\alpha$, then $x\alpha^{-1} \cap (((A\alpha)')\alpha^{-1})' \neq \square$, and hence $x\alpha^{-1} \not\subseteq ((A\alpha)')\alpha^{-1}$. From Lemma 1, we get $x \notin (A\alpha)'$ and therefore $x \in A\alpha$.

The following theorem provides an explicit anti-isomorphism between the lattices $V(\alpha)$ and $V(\alpha^{-1})$. This theorem is a slight modification of a result due to Zaretskii [6]. A sketch of the proof is given.

THEOREM 1. Let α *be an element of* $\mathcal{B}_X$. *The map* ψ : $V(\alpha) \to V(\alpha^{-1})$ *defined by* $(A\alpha)\psi = ((A\alpha)')\alpha^{-1}$ *is a lattice anti-isomorphism.*

Proof. Define $\overline{\psi}$: $V(\alpha^{-1}) \to V(\alpha)$ by $(A\alpha^{-1})\overline{\psi} = ((A\alpha^{-1})')\alpha$. It follows from Lemma 4 that $\psi\overline{\psi}$ is the identity on $V(\alpha)$ and $\overline{\psi}\psi$ is the identity on $V(\alpha^{-1})$.

If $A\alpha \subseteq B\alpha$, then $(A\alpha)' \supseteq (B\alpha)'$. By Lemma 1, we get that $(A\alpha)\psi = ((A\alpha)')\alpha^{-1} \supseteq ((B\alpha)')\alpha^{-1} = (B\alpha)\psi$, so that ψ is an order reversing map.

LEMMA 5. Let β *be in* $\mathcal{B}_X$, ϕ *in* $\mathrm{Aut}(V(\beta))$, *and define* α *by* $(x,y) \in \alpha$ *if and only if* $(x\beta)\phi \supseteq y\beta$. *Then* $\alpha\beta\mathcal{L}\beta$.

Proof. Define a binary relation γ by $(x,y) \in \gamma$ if and only if $(x\beta)\phi^{-1} \supseteq y\beta$. We will prove that $\alpha\beta\mathcal{L}\beta$ by showing that $\beta = \gamma\alpha\beta$.

Suppose $(x,y) \in \beta$. Then $y \in x\beta$, and hence $x\beta$ is non-empty. Since ϕ is an automorphism of $V(\beta)$, it follows that $(A\beta)\phi = \square$ if and only if $A\beta = \square$. Thus $x\beta = (A\beta)\phi$ for some non-empty subset A of X. Hence

$$y \in x\beta = (A\beta)\phi = (\cup_{z\in A} z\beta)\phi = \cup_{z\in A}(z\beta)\phi,$$

so that $y \in (a\beta)\phi$ for some $a \in A$. Now $x\beta \supseteq (a\beta)\phi$ so that $(x\beta)\phi^{-1} \supseteq a\beta$ and hence $(x,a) \in \gamma$ by definition.

In the same manner, since $(a\beta)\phi$ is non-empty it follows that

$$y \in (a\beta)\phi = B\beta = \cup_{w\in B} w\beta$$

for some non-empty subset B of X. Thus, $y \in b\beta$ for some $b \in B$, and hence $(b,y) \in \beta$. Moreover, $(a\beta)\phi \supseteq b\beta$ so that $(a,b) \in \alpha$ by the definition of α. It follows that $\beta \subseteq \gamma\alpha\beta$.

To show the reverse inclusion, let $(x,y) \in \gamma\alpha\beta$. Then there exists $u,v \in X$ such that $(x,u) \in \gamma$, $(u,v) \in \alpha$, and $(v,y) \in \beta$. Now $(x,u) \in \gamma$ implies $(x\beta)\phi^{-1} \supseteq u\beta$, which in turn implies $x\beta \supseteq (u\beta)\phi$. Similarly, $(u,v) \in \alpha$ implies $(u\beta)\phi \supseteq v\beta$, and hence $x\beta \supseteq v\beta$. Now $(v,y) \in \beta$ means $y \in v\beta$ and thus $y \in x\beta$ so that $(x,y) \in \beta$, and hence $\gamma\alpha\beta \subseteq \beta$. It follows that $\gamma\alpha\beta = \beta$, and hence $\alpha\beta\mathcal{L}\beta$.

MAIN RESULTS

The following theorem will make it possible to define a map from the automorphism group of $V(\beta)$ into the dual Schützenberger group of the $\mathcal{H}$-class containing β.

THEOREM 2. Let $\beta \in \mathcal{B}_X$, $\phi \in \mathrm{Aut}(V(\beta))$, *and define* α *by* $(x,y) \in \alpha$ *if and only if* $(x\beta)\phi \supseteq y\beta$. *Then* $\alpha\beta\mathcal{H}\beta$.

Proof. We have $\alpha\beta\mathcal{L}\beta$ from Lemma 5. Let $\overline{\phi} = \psi^{-1}\phi\psi$ where ψ : $V(\beta) \to V(\beta^{-1})$ is the anti-isomorphism of Theorem 1 defined by $(A\beta)\psi = ((A\beta)')\beta^{-1}$. Then $\overline{\phi}$ and $\overline{\phi}^{-1}$ are both in $\mathrm{Aut}(V(\beta^{-1}))$. If δ is defined by $(x,y) \in \delta$ if and only if $(x\beta^{-1})\overline{\phi}^{-1} \supseteq y\beta^{-1}$, then $\delta\beta^{-1}\mathcal{L}\beta^{-1}$ by Lemma 5. Using Lemma 2, we get $\beta\delta^{-1}\mathcal{R}\beta$. To show that $\alpha\beta\mathcal{H}\beta$ it is sufficient to show that $\alpha\beta = \beta\delta^{-1}$.

We first show that $\alpha\beta \subseteq \beta\delta^{-1}$. Let $(x,y) \in \alpha\beta$. Then $(x,u) \in \alpha$ and $(u,y) \in \beta$ for some $u \in X$. Thus,

$$u \in y\beta^{-1} = (A\beta^{-1})\overline{\phi} = (\cup_{z\in A} z\beta^{-1})\overline{\phi} = \cup_{z\in A}(z\beta^{-1})\overline{\phi}$$

for some non-empty subset A of X, so that $u \in (a\beta^{-1})\overline{\phi}$ for some $a \in A$. We now show that $(x,a) \in \beta$. Suppose, on the contrary, that $(x,a) \notin \beta$. From the definition of ψ it follows that $(x\beta)\psi = ((x\beta)')\beta^{-1} \supseteq a\beta^{-1}$. Since ψ is order reversing, we get $x\beta \subseteq (a\beta^{-1})\psi^{-1}$, and hence $(x\beta)\phi \subseteq (a\beta^{-1})\psi^{-1}\phi$. Now $(x,u) \in \alpha$ implies $u\beta \subseteq (x\beta)\phi$, and hence $u\beta \subseteq (a\beta^{-1})\psi^{-1}\phi$, from which it follows that $(u\beta)\psi \supseteq (a\beta^{-1})\psi^{-1}\phi\psi = (a\beta^{-1})\overline{\phi}$. But $u \in (a\beta^{-1})\overline{\phi}$, and hence $u \in (u\beta)\psi = ((u\beta)')\beta^{-1}$, which implies $u\beta \cap (u\beta)' \neq \square$, which is impossible. Hence $(x,a) \in \beta$.

It remains to show that $(a,y) \in \delta^{-1}$. Now $y\beta^{-1} \supseteq (a\beta^{-1})\overline{\phi}$ and hence it follows that $(y\beta^{-1})\overline{\phi}^{-1} \supseteq a\beta^{-1}$. By the definition of δ we have $(y,a) \in \delta$ and hence $(a,y) \in \delta^{-1}$.

We now have $\alpha\beta \subseteq \beta\delta^{-1}$. The proof of the reverse inclusion is now immediate, for $(x,y) \in \beta\delta^{-1}$ implies $(y,x) \in \delta\beta^{-1} \subseteq \beta^{-1}\alpha^{-1}$ by the first part of the proof (with β replaced by β^{-1}, ϕ replaced by $\overline{\phi}^{-1}$, and α replaced by δ), hence $\alpha\beta = \beta\delta^{-1}$.

The following lemma is a special case of a theorem in Clifford and Preston [3, Theorem 2.3].

LEMMA 6. Let H be an $\mathcal{H}$-class of a semigroup S. If $t \in S$ and $tg \in H$ for some $g \in H$, then $tH \subseteq H$.

If α is $\mathcal{H}$-related to β, then $V(\alpha) = V(\beta)$ [6]. Thus, if the lattice corresponding to an $\mathcal{H}$-class is interpreted as the row lattice of any element of the $\mathcal{H}$-class, then the following theorem shows that the Schützenberger group of an $\mathcal{H}$-class of $\mathcal{B}_X$ is isomorphic to the automorphism group of the lattice corresponding to the $\mathcal{H}$-class. This is an extension of a result of Zaretskii [6, Theorem 3.9].

THEOREM 3. Let X be a non-empty set. If H is an $\mathcal{H}$-class of $\mathcal{B}_X$ and $\beta \in H$, then the automorphism group of $V(\beta)$ is isomorphic to the Schützenberger group of H.

Proof. We will construct an anti-isomorphism from $Aut(V(\beta))$ onto the dual Schützenberger group $\Gamma'(H)$.

For $\alpha \in \mathcal{B}_X$, let λ_α denote inner left translation by α restricted to H; that is, $x\lambda_\alpha = \alpha x$ for all $x \in H$. The dual Schützenberger group of H is the set $\Gamma'(H)$ of all λ_α such that $\alpha H \subseteq H$.

Define a map $\Lambda : Aut(V(\beta)) \to \Gamma'(H)$ by $\phi\Lambda = \lambda_\alpha$ where α is the binary relation determined by $(x,y) \in \alpha$ if and only if $(x\beta)\phi \supseteq y\beta$. By Theorem 2, $\alpha\beta \in H$. From Lemma 6, it follows that $\alpha H \subseteq H$, and hence λ_α is indeed an element of $\Gamma'(H)$.

Define $\overline{\Lambda} : \Gamma'(H) \to Aut(V(\beta))$ by $(\lambda_\alpha)\overline{\Lambda} = \phi$ where $(A\beta)\phi = A\alpha\beta$ for all $A\beta \in V(\beta)$. We will show that Λ is one-to-one and onto by showing that $\overline{\Lambda}$ is a two-sided inverse for Λ.

We need to show that $\phi \in AutV(\beta))$. To show that ϕ is well-defined, we note that $\alpha\beta = \beta\rho$ for some $\rho \in \mathcal{B}_X$ since $\alpha\beta\mathcal{H}\beta$. Thus, if $A\beta = B\beta$, then $A\beta\rho = B\beta\rho$, and hence

$$(A\beta)\phi = A\alpha\beta = A\beta\rho = B\beta\rho = B\alpha\beta = (B\beta)\phi.$$

There is also a relation μ such that $\alpha\beta\mu = \beta$. Hence, if $(A\beta)\phi = (B\beta)\phi$, then $A\alpha\beta = B\alpha\beta$, which implies $A\beta = A\alpha\beta\mu = B\alpha\beta\mu = B\beta$. It follows that ϕ is one-to-one.

Since $\beta = \sigma\alpha\beta$ for some relation σ, we have, for any $A\beta \in V(\beta)$, $[(A\sigma)\beta]\phi = (A\sigma)\alpha\beta = A\sigma\alpha\beta = A\beta$, so that ϕ maps $V(\beta)$ onto $V(\beta)$.

It is straightforward to show that $A\beta \subseteq B\beta$ if and only if $(A\beta)\phi \subseteq (B\beta)\phi$, and hence that $\phi \in Aut(V(\beta))$.

Let $\theta \in Aut(V(\beta))$ and let $\phi = (\theta)\Lambda\overline{\Lambda}$ where $\theta\Lambda = \lambda_\alpha$. Let $A\beta \in V(\beta)$. Then $(A\beta)\phi = A\alpha\beta$. If $b \in (A\beta)\theta$, then $b \in (a\beta)\theta$ for some $a \in A$. Thus there exists $c \in X$ such that $b \in c\beta \subseteq (a\beta)\theta$. Hence $(a,c) \in \alpha$ (by definition of α), $(c,b) \in \beta$ and thus $b \in A\alpha\beta$. Consequently, $(A\beta)\theta \subseteq A\alpha\beta$.

Conversely, let $x \in A\alpha\beta$. Then $(a,x) \in \alpha\beta$ for some $a \in A$, so that $(a,u) \in \alpha$ and $(u,x) \in \beta$ for some $u \in X$. Hence $(a\beta)\theta \supseteq u\beta$ and $x \in u\beta$ implies $x \in (a\beta)\theta \subseteq (A\beta)\theta$. Thus, $(A\beta)\theta = (A\beta)\phi$ for every $A\beta \in V(\beta)$ so that $\theta = \phi$. It follows that $\Lambda\overline{\Lambda}$ is the identity on $Aut(V(\beta))$.

Let $\lambda_\alpha \in \Gamma'(H)$, let $\lambda_\gamma = \lambda_\alpha\overline{\Lambda}\Lambda$ and let $\phi = \lambda_\alpha\overline{\Lambda}$. Then $(x,y) \in \gamma$ if and only if $(x\beta)\phi \supseteq y\beta$; that is, if and only if $x\alpha\beta \supseteq y\beta$. Suppose $(x,y) \in \alpha\beta$. Then $(x,u) \in \alpha$ and $(x,y) \in \beta$ for some $u \in X$. Thus, $u \in x\alpha$ implies $u\beta \subseteq x\alpha\beta$ and therefore $(x,u) \in \gamma$. It follows that $(x,y) \in \gamma\beta$ and hence $\alpha\beta \subseteq \gamma\beta$.

Conversely, if $(x,y) \in \gamma\beta$, then $(x,v) \in \gamma$ and $(v,y) \in \beta$ for some $v \in X$. Now $(x,v) \in \gamma$ implies $v\beta \subseteq x\alpha\beta$, and since $y \in v\beta$, it follows that $y \in x\alpha\beta$; that is, $(x,y) \in \alpha\beta$. Therefore, $\lambda_\alpha = \lambda_\gamma$ and hence $\overline{\Lambda}\Lambda$ is the identity on $\Gamma'(H)$.

Since $\lambda_{\alpha\delta} = \lambda_\delta\lambda_\alpha$, it is not difficult to show that $\overline{\Lambda}$ is an anti-homomorphism; that is $(\lambda_\alpha\lambda_\delta)\overline{\Lambda} = (\lambda_\delta\overline{\Lambda})(\lambda_\alpha\overline{\Lambda})$. It then follows that Λ is also an anti-homomorphism and therefore $Aut(V(\beta))$ is anti-isomorphic to $\Gamma'(H)$. Since $\Gamma'(H)$ is anti-isomorphic to $\Gamma(H)$, it follows that $Aut(V(\beta))$ and $\Gamma(H)$ are isomorphic and the proof is complete.

For $\beta \in \mathcal{B}_X$, let $T_\beta = \{\sigma \in G_X : \sigma\beta = \beta\rho$ for some $\rho \in G_X$ and $x\sigma = x$ if $x\beta = \square\}$, where G_X is the symmetric group on X. It is straightforward to show that T_β is a subgroup of G_X.

The following theorem gives another characterization of the Schützenberger group $\Gamma(H_\beta)$ if β is a reduced relation. A special case of this theorem is due to Montague and Plemmons [4, Theorem 3.5].

THEOREM 4. *Let β be a reduced relation in $\mathcal{B}_X$. Then $\Gamma(H_\beta)$ is isomorphic to the subgroup T_β of G_X.*

Proof. To prove that T_β is isomorphic to $\Gamma(H_\beta)$ we show that T_β is isomorphic to $\mathrm{Aut}(V(\beta))$, then apply Theorem 3. Define a map $\Phi : T_\beta \to \mathrm{Aut}(V(\beta))$ by $\sigma\Phi = \phi_\sigma$ where $(A\beta)\phi_\sigma = A\sigma\beta$. It is routine to show that $\phi_\sigma \in \mathrm{Aut}(V(\beta))$, that Φ is a homomorphism, and that Φ is one-to-one.

To show that Φ is onto, let $\phi \in \mathrm{Aut}(V(\beta))$. Define σ by

$$x\sigma = \begin{cases} y & \text{if } x\beta \neq \square \text{ and } (x\beta)\phi = y\beta \\ x & \text{if } x\beta = \square. \end{cases}$$

Since β is a reduced relation and since ϕ maps the set of join-irreducible elements of $V(\beta)$ onto itself, it follows that if $x\beta \neq \square$, then $(x\beta)\phi = y\beta$ for some $y \in X$. Furthermore, if $x\beta$ and $y\beta$ are non-empty and $(x\beta)\phi = (y\beta)\phi$, then $x\beta = y\beta$, but β is reduced, and hence $x = y$. It follows that $\sigma \in G_X$.

To show that $\sigma \in T_\beta$ it suffices to show that $\sigma\beta = \beta\rho$ for some $\rho \in G_X$. We first show that $\sigma\beta\mathcal{H}\beta$. Let α be defined as in Theorem 2; that is, $(x,y) \in \alpha$ if and only if $(x\beta)\phi \supseteq y\beta$. We show that $\sigma\beta = \alpha\beta$ and then apply Theorem 2 to conclude that $\alpha\beta\mathcal{H}\beta$. Certainly $\sigma \subseteq \alpha$, and hence $\sigma\beta \subseteq \alpha\beta$.

On the other hand, suppose $(x,y) \in \alpha\beta$. Then there exists $u \in X$ such that $(x,u) \in \alpha$ and $(u,y) \in \beta$. This means that $y \in u\beta \subseteq (x\beta)\phi$. It follows that $x\beta \neq \square$ so that $(x\beta)\phi = z\beta$ for some $z \in X$. But then $y \in z\beta$ and $(x,z) \in \sigma$, which implies $(x,y) \in \sigma\beta$. Hence $\sigma\beta = \alpha\beta$.

It follows from a theorem of Plemmons and West that $\sigma\beta = \beta\rho$ for some $\rho \in G_X$ [5, Theorem 1.9]. Hence $\sigma \in T_\beta$.

To show that Φ is onto, it remains to show that $\sigma\Phi = \phi$. Let $x \in X$. If $x\beta = \square$, then $(x\beta)(\sigma\Phi) = (x\beta)\phi_\sigma = x\sigma\beta = x\beta = \square$ and hence $(x\beta)\phi = \square$. Suppose $x\beta \neq \square$. Then $x\sigma = y$ where $(x\beta)\phi = y\beta$. But then $(x\beta)\phi_\sigma = x\sigma\beta = y\beta = (x\beta)\phi$. Therefore $\phi = \phi_\sigma = \sigma\Phi$.

COROLLARY. *Suppose X is finite and let $\beta \in \mathcal{B}_X$. Then there exists $\alpha \in \mathcal{B}_X$ such that $\alpha\mathcal{D}\beta$ and $\Gamma(H_\beta) \cong T_\alpha$.*

Proof. Since X is finite, the $\mathcal{D}$-class D_β contains a reduced relation α [4]. Thus, $\Gamma(H_\beta) \cong \Gamma(H_\alpha) \cong T_\alpha$.

REFERENCES

[1] Birkhoff, G., *Lattice theory*, A.M.S. Colloquium Publ., Vol. 25, 2nd ed., Providence, 1948.

[2] Brandon, R. L., K. K.-H. Butler, D. W. Hardy and G. Markowsky, *Cardinalities of $\mathcal{D}$-classes in $\mathcal{B}_n$*, Semigroup Forum 4(1972), 341-344.

[3] Clifford, A. H. and G. B. Preston, *The algebraic theory of semigroups*, A.M.S. Surveys, Vol. I, Providence, 1961.

[4] Montague, J. S. and R. J. Plemmons, *Maximal subgroups of the semigroup of relations*, J. of Algebra 13(1969), 575-587.

[5] Plemmons, R. J. and M. T. West, *On the semigroup of binary relations*, Pac. J. of Math. 35(1970), 743-753.

[6] Zaretskii, K. A., *The semigroup of binary relations*, Mat. Sbornik 61(1963), 291-305.

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