AN OVERVIEW OF THE POSET OF IRREDUCIBLES

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1 Introduction

An interesting fact that is of great practical importance is that finite lattices have an associate poset, called the *poset of irreducibles* that acts much like the basis of a vector space. The poset of irreducibles of a finite lattice provides a compact representation of the lattice from which many of the properties of the lattice can be deduced easily. This paper is dedicated to explaining the poset of irreducibles and providing some examples of its usefulness. Proofs are omitted except for the very simple ones, but all results can be found in the references located at the end of this paper. These results can be extended to infinite lattices, but we will not discuss such extensions here. The interested reader is invited to read [4] and [5] for more details. [12] provides additional historical and motivational material, which might be of interest to the reader.

The poset of irreducibles generalizes the construction used by Garrett Birkhoff [2] to provide a representation for finite distributive lattices. Birkhoff proved that a distributive lattice, L, is isomorphic to the lattice of all closed from below subsets of the poset consisting of the join-irreducible elements of L in the induced order. An interesting extension of this result is that the connected components of the poset of join-irreducibles correspond to the posets of join-irreducibles of the Cartesian factors of a lattice and that the automorphism group of the poset of join-irreducibles is isomorphic to the automorphism group of the lattice. Since in a distributive lattices the poset of meet-irreducibles is isomorphic to the poset of join-irreducibles. It is sufficient to work with either the join-irreducibles or meet-irreducibles. Figure 1 illustrates Birkhoff's Theorem.

Definition 1 *A* join-irreducible *element, j, of a lattice, L, is has the property that* $j = \sup S$*, where S is a subset of L, implies that j is in S.* □

The bottom element of a lattice is never join-irreducible since it is the join of the empty set. There is a dual definition of meet-irreducible.

The focus on irreducibles is a key aspect of a combinatorial approach to lattice theory. In this approach, one focuses on things such as the Hasse diagram of a lattice rather than on algebraic identities satisfied by elements of

the lattice. Especially satisfying, are results that connect algebraic properties to combinatorial properties.

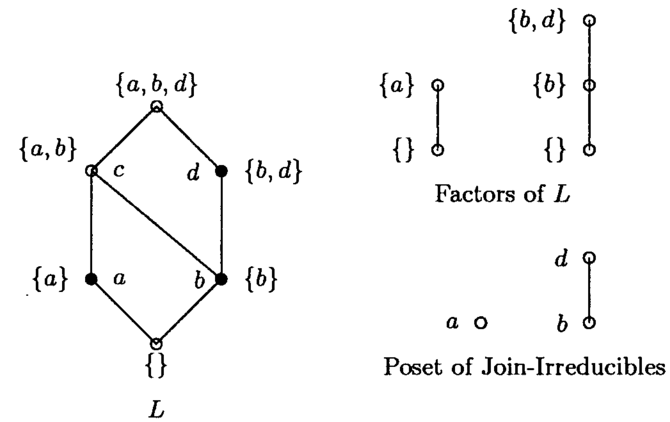

Figure 1. An Illustration of Birkhoff's Theorem

One example of such a theorem is a result that I discovered in my thesis ([4]), but which I later found had been discovered a decade earlier by Avann ([1]). The result is the following:

Theorem 1 *(Avann, Markowsky)*
A finite lattice is distributive if and only if

1. *The number of meet-irreducibles equals the length of the lattice.*
2. *The number of join-irreducibles equals the length of the lattice.*
3. *The lattice satisfies the Jordan-Dedekind chain condition.* □

Figure 2 shows three simple lattices that illustrate the graphical test for distributivity. In the first case, the lattice satisfies the Jordan-Dedekind chain condition, but both the number of join-irreducibles and the number of meet-irreducibles are greater than the length of the lattice. In the second case, the lattice has more join-irreducibles than either the number of meet-irreducibles or the length of the lattice. In the third case, the lattice satisfies all the requirements and is distributive.

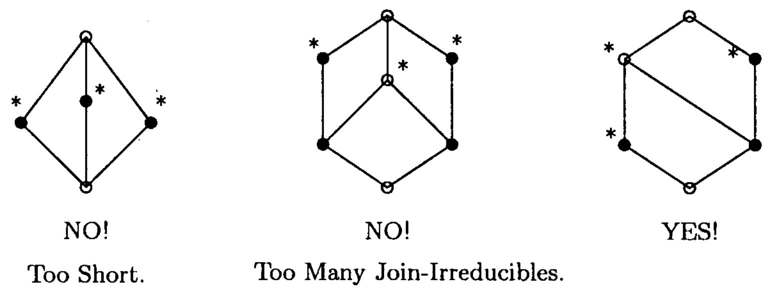

Figure 2. Examples of the Graphical Test for Distributivity

2 The Poset of Irreducibles

It seems clear that for general lattices both the join-irreducible and meet-irreducible elements need to be considered. Since elements can be both join-irreducible and meet-irreducible, it seems reasonable to consider a bipartite graph where an element can appear twice if necessary. One natural construction is to put the meet-irreducibles in a row over a row of join-irreducibles and connect an element in the top row to an element in the bottom row if the top element is $\geq$ the element in the bottom row. Interestingly enough, a more useful construction is to connect the top element to the bottom element iff the top element is $\not\geq$ the bottom element. The big advantage of the second construction over the first, is that the Cartesian factors of a lattice can be read directly from the associated poset, because the connected components of the poset (when the poset is considered a graph) correspond to the Cartesian factors of the lattice.

Figure 3 shows the lattice M3, the induced order on the irreducibles, a bipartite graph using the induced order to relate the two rows of irreducibles, and finally the complementary order on the irreducibles. The induced order has the undesirable property that it splits into 3 connected components while the lattice does not have direct factors. The complementary order, on the other hand, has only a single connected component.

Figure 4 shows the same constructions applied to the Boolean algebra with 3 atoms. Note that in this case the lattice has 3 direct factors, while

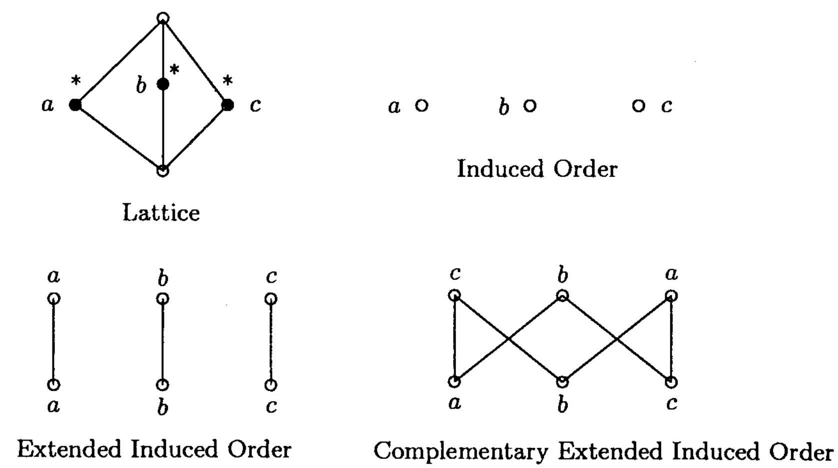

Figure 3. Illustration of the Poset of Irreducibles and Related Constructions

the bipartite directed graph (bidigraph for short) that uses the induced order consists of only one connected component. On the other hand, the bidigraph derived from the complementary order has 3 connected components.

Definition 2 *Given a finite lattice L, the poset of irreducibles,* P(L), *is the poset formed by putting all the join-irreducibles of P(L) in a row and then placing all the meet-irreducibles in a row above the join-irreducibles, and ordering them as follows. In P(L), a meet-irreducible element, m, is above a join-irreducible element, j, iff $m \neq j$ in L.*

The Poset of Irreducibles was introduced in my thesis in 1972-73 [4], and developed in a series of papers published from 1973 through 1994. In 1982 Wille [14]in a paper entitled *Restructuring Lattice Theory* introduced the terms concept lattices and context. A context is the same thing as the bipartite poset of irreducibles discussed above, but with the induced order. As noted earlier this construction does not make evident the Cartesian factorization of a lattice, but is simply the dual of the Poset of Irreducibles construction. Even though Wille and his school have been aware of my work on the Poset of Irreducibles since 1973 they have not acknowledged it in their work.

The technique for recovering a lattice from its poset of irreducibles is

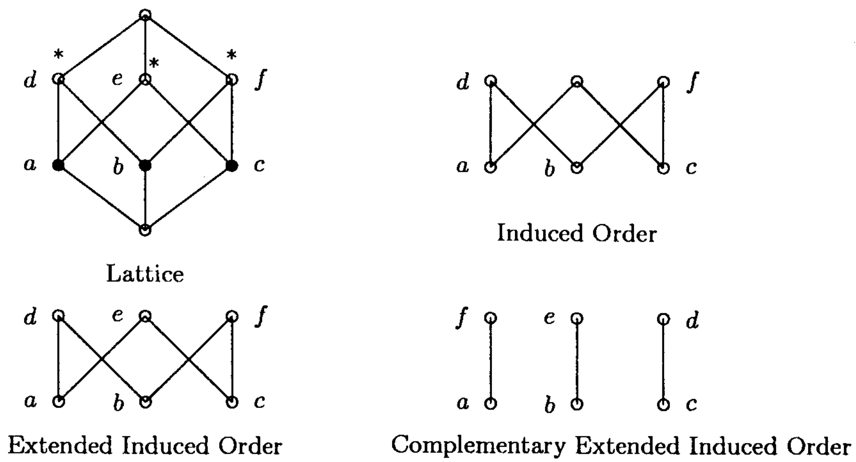

Figure 4. Illustration of the Poset of Irreducibles and Related Constructions

fairly straightforward:

1. For each element on the bottom row, form the set of all elements in the top row that are connected to it.

2. The set of unions of all such sets (we include the empty set as the empty union) ordered by set inclusion is isomorphic to the original lattice.

Figure 5 shows the basic reconstruction process. Note that we use the abbreviation Rep to represent the set of meet-irreducibles linked to a particular join-irreducible on the bottom row.

The calculations of the three Reps in Figure 5 is straightforward as is the construction of all unions. It is easy to see that we recover the original lattice in this way. Figure 6 shows the same construction for the 3-atom Boolean algebra.

Reconstruction can also be done from different perspectives such as Galois connections and the lattice of maximal antichains. For details see [6]. There has been some interesting work done in the area of lattice reconstruction by Morvan and Nourine [13], and by Jourdan, Rampon and Jard [3].

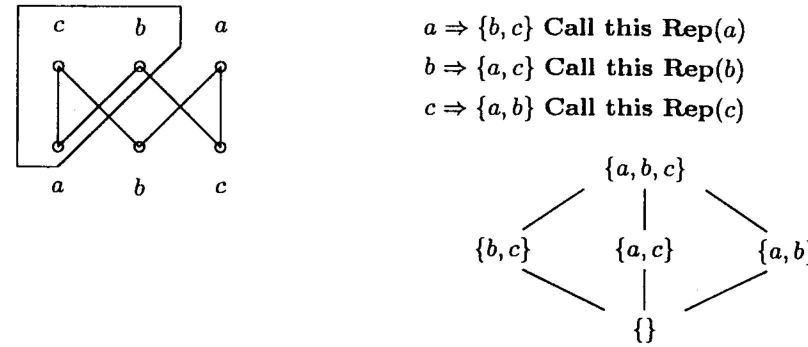

Figure 5. Reconstructing the Lattice

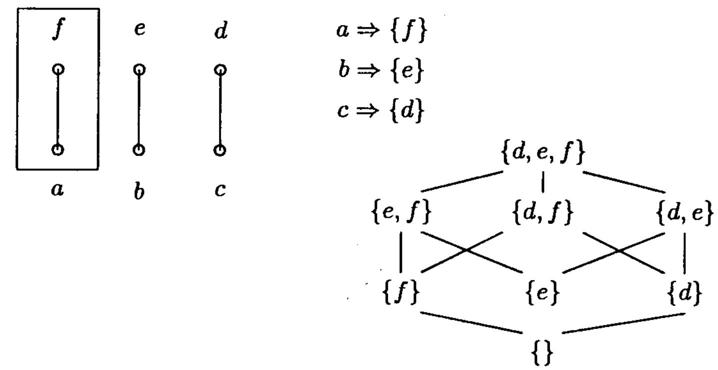

Figure 6. Reconstructing the Boolean Algebra

To summarize the preceding discussion we note that the Poset of Irreducibles of a lattice L, denoted by P(L), is possibly a compact representation of a lattice. In my thesis I proved a generalized form of the following result which can be used with some infinite lattices.

Theorem 2 *(Markowsky [4] and [6]) Given a finite lattice L. Its poset of irreducibles has the following properties:*

1. *L can be easily reconstructed from P(L) using the union construction.*

2. *The connected components of P(L) are the posets of irreducibles of the direct factor lattices whose product is L.*

3. *The group of all order preserving automorphisms of L is isomorphic to the group of all order preserving automorphisms of P(L).* □

In general, P(L) is significantly smaller than L. In the case of Boolean algebras, P(L) is exponentially smaller than L.

Throughout this we will use *J(L)* to denote the set of join-irreducibles of a lattice L, and *M(L)* to denote the set of meet-irreducibles of L.

Let's consider one more example. Figure 7 shows that P(L) makes it easy to spot the fact that a lattice can be factored directly and that the factors can be computed directly from the components of P(L).

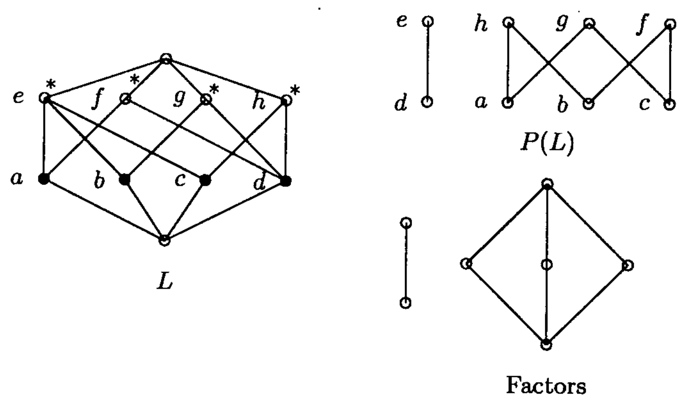

Figure 7. A more complicated example of the Poset of Irreducibles

Notice that since the poset of irreducibles of each Cartesian component of the lattice is given by a connected component of the poset of irreducibles of the lattice, the factors are themselves not further reducible.

Another interesting question to consider is which bidigraphs (bipartite digraphs) can be P(L) for some lattice L. To give this condition we just need to extend the definition of Rep for any bidigraph. In particular, if S is set of nodes of the bidigraph, G = (X, Y, Arcs), let Rep(S) = all nodes that are

linked in G to some node in S. Now we can characterize which bidigraphs are P(L) for some lattice L .

Theorem 3 *(Markowsky [4, 6]) A bidigraph G is P(L) for some lattice L iff the following condition holds for each node n:*
Rep(n) = Rep(S) can only happen if n is in S. □

3 Applications

Looking at lattices from the point of view of their posets of irreducibles, provides another approach to solving problems and better understanding the features of the lattices in question. The following subsections briefly sketch some of the instances where focusing on the poset of irreducibles has led to some key insights.

3.1 Locally Distributive Lattices

Earlier, a simple test for distributivity was mentioned. A slight modification of this result provides a simple characterization of locally distributive lattices. In particular, we get the following theorem.

Theorem 4 *(Avann [1], Greene and Markowsky [10]) A finite lattice is upper (lower) locally distributive iff*

1. *It is Jordan-Dedekind*

2. *Its meet-rank (join-rank) = its length* □

Note that the *meet-rank (join-rank)* of a lattice is simply the number of meet-irreducible (join-irreducible) elements in the lattice.

3.2 Factor-Union Representation

Associated with the poset of irreducibles are some theorems that provide some information about mappings between lattices. A very fundamental theorem is the following.

Theorem 5 *(Markowsky [4, 5, 6, 9])*

1. *If $f : L_1 \to L_2$ is join-preserving then $|M(L_1)| \leq |M(L_2)|$.*

2. *The map $f : L \to 2^{M(L)}$ given by: $f(a) = \{m \epsilon M(L) | a \leq m\}$ preserves sups.*

3. *The mapping f in the preceding item is optimal in the sense that no smaller Boolean algebra can be found to represent L with unions representing sups.*

□

The last part of the preceding theorem demonstrates that the poset of irreducibles is the smallest construction that can be used to represent the lattice using unions of sets.

The preceding theorem can be used to develop an algorithm for determining if a genetic system can be described as a union of different factors. Systems that can be described in such a manner are called *factor-union systems.*

Suppose that a group of individuals displays genetic variations, and you would like to understand how genes can carry traits. One simple model of such behavior represents traits as being made from unions of simpler traits. For example, consider a simplified eye-color model in which there are only blue eyes and brown eyes. Further, suppose that brown eye genes are dominant over blue eye genes.

Recall that a *phenotype* is a type that can be objectively recognized such as brown-eye vs. blue-eye. Also, a genotype is a particular combination of genes. In general, multiple genotypes might produce the same phenotype. In the system under discussion the phenotype of having brown eyes consists of the 3 genotypes: (brown, brown), (brown, blue), and (blue, brown). On the other hand, the genotype (blue, blue) is the only one in the blue eye phenotype. The ordered pairs represent the combination of genes that the individual gets from each parent. A simple factor-union model for this eye color system is the following: assume that having blue eyes is the default and requires no particular trait, whereas a brown eye-color gene contains a single factor x, which colors eyes brown if it is present. In this case, the three genotypes (brown, brown), (brown, blue) and (blue, brown) will produce an individual with brown eyes, while (blue,blue) will produce a blue-eyed individual.

To determine whether a factor-union representation in possible for some system of phenotypes, we must order phenotypes based on the assumption that the system in question is a factor-union system. If it is indeed a factor-union system, then the algorithm will eventually produce a lattice in which unions represent sups. If the system in question is not a factor-union system, the algorithm eventually produce a cycle of distinct elements such a ¡ b ¡ a, which is impossible in a poset. For the details of this construction see [9].

If no cycles appear while the order is being completed, then the algorithm produces a lattice and any join-representation of that lattice is a factor-union representation of the original system. By the preceding theorem, a minimal factor-union representation is constructed using the meet-irreducibles of the generated lattice. Note that the minimal representation need not be the correct biological model. In fact, just because a factor-union representation can be found for a genetic system one cannot simply assume that it is the correct explanation. This determination must be made on a biological basis, but the results here suggest some starting points.

The results in [9] apply to multi-locus systems as well as single-locus systems.

3.3 Subprojective Lattices and Projective Geometry

A variety of people have developed axioms systems for projective geometries. Initially, all of the axiom systems proposed contained a numerical parameter, and hence were unsuitable for infinite dimensional projective geometries. Basing their work on the poset of irreducibles, Markowsky and Petrich [7] produced a purely point and hyperplane, numerical-parameter-free, self-dual axiomatization of subprojective lattices. In finite dimensions, subprojective lattices are also projective, so this axiomatization gives a parameter-free parametrization of finite dimensional projective geometries. This work also provided conditions under which subprojective lattices became projective.

3.4 Extremal Lattices

The Jordan-Dedekind chain condition is a strong condition to require. As a result of the prompting of Garrett Birkhoff, I investigated what can be said in the absence of the Jordan-Dedekind chain condition, and focused in particular on the case where the length of a lattice matched its join-rank and/or its meet-rank. It is clear that every element that covers another must have at least one additional join-irreducible below it and at least one fewer meet-irreducible above it than the element it covers. Thus, for any lattice L, length(L) $\leq |J(L)|, |M(L)|$.

Definition 3 *1. A lattice, L, is called* join-extremal *iff length(L)* $= |J(L)|$.

2. *A lattice, L, is called* meet-extremal *iff length(L)* $= |M(L)|$.

3. *A lattice, L, is called* extremal *iff length(L)* $= |J(L)| = |M(L)|$. □

The various types of extremal lattices have many interesting properties. One simple property is given by the following theorem.

Theorem 6 *(Markowsky [10]). A Cartesian product of lattices is (join-, meet-) extremal iff each factor is (join-, meet-) extremal.* □

The term *p-extremal* (p = the empty string, "join", "meet") is used to refer to any of the three types of extremal lattices. Just be sure to make the same substitution for the prefix p in the same context. P-extremal lattices have many interesting properties and generalize decompositions of finite Boolean algebras. An interesting fact is that ideals of join-extremal lattices are join-extremal, and dual ideals of meet-extremal lattices are meet-extremal.

Another interesting fact about p-extremal lattices is that they cannot be categorized algebraically. Furthermore, the family of p-extremal lattices includes many interesting lattice families including distributive lattices, locally distributive lattice, and Tamari associativity lattices (see below). The first step to proving some of these results is to characterize the posets of irreducibles of extremal lattices (we can also do this for p-extremal lattices in general).

Theorem 7 *(Markowsky [10]) A bidigraph (X, Y, Arcs) is P(L) for an extremal lattice iff:*

1. $|X| = |Y| = n$.
2. *You can number X and Y from 1 to n such that*
   (a) (x_i, y_i) *is an arc for all i.*
   (b) *if* (x_i, y_j) *is an arc,* $i \leq j$. □

Using the characterization of P(L) for extremal L leads immediately to the following results.

Theorem 8 *(Markowsky [10]) Any finite lattice is isomorphic to an interval of some finite extremal lattice.* □

Corollary 9 *(Markowsky [10]) Extremal lattices cannot be characterized algebraically.* □

Of special importance when working with p-extremal lattices are the *coprime* and *prime* elements.

Definition 4 1. *An element* $a \neq O$ *in L is called* coprime *if for all x and y in L,* $x \vee y \geq a$ *implies that* $x \geq a$ *or* $y \geq a$.

2. *An element aneqI in L is called* prime *if for all x and y in L,* $x \wedge y \leq a$ *implies that* $x \leq a$ *or* $y \leq a$. □

Coprimes are special kinds of join-irreducibles, while primes are special kinds of meet-irreducibles. The following three result is a straightforward consequence of the above definitions and is found in [10].

Theorem 10 *The following are equivalent*

1. *L is distributive.*

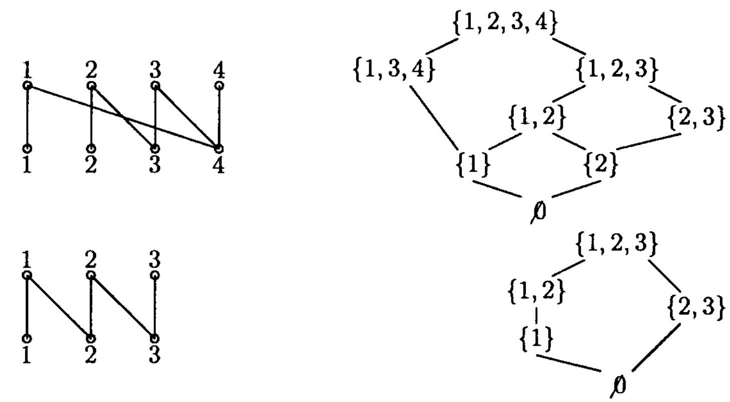

Figure 8. Some Posets of Irreducibles of Extremal Lattices Numbered

2. *All join-irreducibles are coprime.*
3. *All meet-irreducibles are prime.* □

The next result is a bit surprising is a generalization of the fact that in a distributive lattice the poset of join-irreducibles in the induced order is isomorphic to the poset of meet-irreducibles.

Theorem 11 *(Markowsky [10]) In any lattice the subposet of coprimes is isomorphic to the subposet of primes.* □

Corollary 12 *In a distributive lattice J(L) is isomorphic to M(L).* □

The existence of primes and coprimes in lattices is of great significance because it permits you to decompose the lattice into simpler lattices. Of crucial importance is the fact that extremal lattices must contain at least one prime and at least one coprime, which can be used to decompose them. Details on these mappings can be found in [10] and [12]. A key result that summarizes the decomposition properties is the following theorem.

Theorem 13 *(Markowsky [10]) Let L be an extremal lattice. Then L has an atom a that is coprime and a matching prime b such that the intervals* $A = [a, I]$ *and* $B = [O, b]$ *partition L. Let the mapping* $g : B \to B$ *be given by* $g(x) = x \vee a$. *Then the following are true:*

1. *g is injective and for all x, g(x) covers x.*

2. *A is extremal.*

3. $M(L) - M(A) = b$.

4. *Length*$(A) =$ *Length*$(L) - 1$.

5. $J(A) \subseteq J(L) \cup (a \vee J(L))$.

6. *B is join-extremal.* □

It is relatively easy to compute P(A) and P(B) from P(L). Of course, a dual theorem holds for a coatom that is prime. The numbering that exists for extremal lattices can be used to derive an alternative characterization of distributive lattices. For details see [10].

3.5 Tamari Associativity Lattices

Tamari associativity lattices are the lattices that result when you take expressions in n+1 variables and a single binary operator * and you order them as follows. One expression covers another if it can be derived by moving parentheses to the left using associativity. Thus, $(a * b) * c \prec a * (b * c)$. It is a non-trivial fact that this covering relation puts a lattice structure on the expressions. The family of lattices that results for all n is called the family of Tamari associativity lattices.

Figure 9 shows some of the smaller Tamari lattices and indicates various coprime/prime decompositions and some of the relations between consecutive members in the family.

M. K. Bennett and G. Birkhoff determined the posets of irreducibles for the Tamari lattices. It is natural to consider the posets of irreducibles of Tamari lattices. The results are summarized in the following theorem.

Theorem 14 *(Markowsky [10])*

1. *Tamari lattices are extremal.*

2. *The coprimes are exactly the atoms, the primes are exactly the coatoms.*

3. *The longest maximal chain has length n(n-1)/2 and the shortest has length n-1.*

4. *Tamari lattices are self-dual.*

5. T_n *has a coprime/prime decomposition such that B is isomorphic to* T_{n-1}*, and the corresponding A is extremal.*

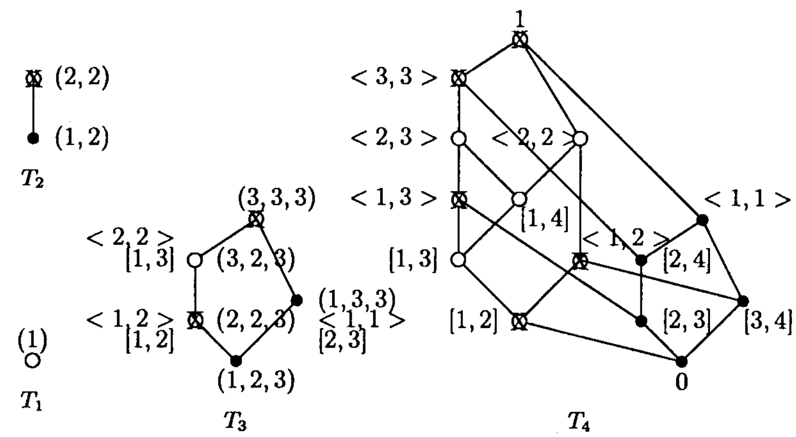

Figure 9. The Decomposition of Tamari Lattices

6. B_n*, the Boolean algebra on n atoms, is a retract of* T_{n+1}.

7. *Every distributive lattice of length n is a sublattice of* T_{n+1}. □

3.6 Permutation Lattices

Permutations can be ordered using a covering relationship similar to the one described for the Tamari lattices. [11] presents the structure of the $P(S_n)$ where S_n is the permutation group on n elements. The key results on the structure of $P(S_n)$ are summarized in the following theorem.

Theorem 15 *(Markowsky [11])*

1. *The join-irreducibles of* S_n *correspond to pairs of subsets of 1,...n,* (A, B)*, such that A and B are complements and A is not of the form 1,...,i for any i.*

2. *The meet-irreducibles of* S_n *correspond to pairs of subsets of 1,...,n,* (C, D)*, such that C and D are complements and D is not of the form 1,...,i for any i.*

3. A join-irreducible, (A, B), of S_n is connected to a meet-irreducible, (C, D), of S_n in $P(S_n)$ iff $\max(A \cap D) > \min(B \cap C)$. □

There are many additional properties of S_n that are derived in [11] that we will not discuss further here.

3.7 Additional Applications

There are many additional applications for the ideas presented in this paper. One application is to check a poset to see whether it is a lattice or not. The idea is to assume that it is a lattice and to construct its poset of irreducibles. One can test whether the resulting bidigraph satisfies the conditions for being a poset of irreducibles of a lattice and whether one reconstructs the original poset from the supposed poset of irreducibles.

Many results about concept lattices are results about the poset of irreducibles of a lattice, the results in that area provide an example of the power of this approach. Furthermore, as noted earlier there are many decompositions and representations that can be applied to lattices. This is an area that deserves further study. For some ideas in this area see [8].

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